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Coincidence and Fixed Points of Nonself Maps using Generalized T-Weak Commutativity

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Abstract: *The purpose of this paper is to introduced the concept of sequential T-weak commutativity and generalized T-weak commutativity, which is an extension of T-weak commutativity defined by Kamran for hybrid pair of mappings. We prove that this concept is equivalent to generalized compatibility of type(N) at coincidence points recently introduced by us and we have shown that compatibility of type (N) is more general than (IT)-commutativity introduced by Singh and Mishra. Using generalized T-weak commutativity we also extend a result of Pathak and Mishra who have indicated that a result of Chang admits a counter example and presented a corrected version of the result. We have also shown that the corrected version of the result of Chang is also derivable from our results. We thus extend and generalize many known results in this way.*

Keywords: *Coincidence and fixed point, Hausdorff metric, generalized compatibility of type(N), generalized T-weak commuting mappings*.

1. Introduction

The study of fixed points of non-self multi-valued and single valued contractions (Hybrid contractions) is a new development in the domain of contractive type multi-valued theory (see, for instance [1,2, 3, 6, 7, 9, 11, 13, 14, 15, 23-26]). Recently, in an attempt to improve/ generalize certain results of Sastry, Naidu and Prasad, [18] and Naidu [15], Chang [3] obtained some fixed point theorems for hybrid of single valued and multi-valued mappings, however as indicated by Pathak and Mishra [17], his main theorem admits a counter example. Pathak and Mishra[17] have also suggested modification and presented a corrected version in more general way using sequential commutativity of hybrid pair of mappings. Our main purpose in this paper is to prove the concept of sequential T-weak commutativity and generalized T-weak commutativity introduced by us in this paper is more general than sequential commutativity introduced by Pathak and Mishra [17]. We also suggest a restriction which must be imposed for defining concept of sequential commutativity.

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2. Preliminaries

We generally follow the definitions and notations used in [1, 2]. Given a metric space (X, d), let (C(X), H), (CB(X), H), and (CL(X), H) denote respectively the hyperspaces of nonempty compact, nonempty closed bounded, and nonempty closed subsets of X, where *H* is the Hausdorff metric induced by *d*. The space (CL(X), H) contains the other two spaces. Throughout, d(A, B) will denote the ordinary distance between nonempty subsets A and B of X while d(x, B) stands for d(A, B) when A = {x}, the singleton set. For any $A \subset X$, $\delta(A)$ will denote the boundary of A. For details of hyperspaces one may refer to [14].

Following Hadzic and Gajic [4] and pant [16], Singh and Mishra [24] have introduced the notion of R-weak commutativity of a hybrid pair of single-valued and multivalued maps, as follows,

Definition 1 : Let K be a nonempty subset of a metric space X, $T : K \to X$ and $F : K \to CL(X)$. Then T and F will be called pointwise R-weakly commuting on K if given $x \in K$ and $Tx \in K$, there exists $R > 0$ such that,

$$d(Ty, FTx) \le Rd(Fx, Tx) \text{ for each } y \in K \cap Fx. \qquad (1)$$

Maps T and F will be called R-weakly commuting on K if for each $x \in K$, $Tx \in K$ and (1) holds for some $R > 0$.

If R = 1, they get the definition of weak commutativity of F and T on K due to Hadzic and Gajic [4] (see [1, 2]). If $F : X \to X$ and $T : X \to X$ then they get the definition of point-wise R-weak commutativity and R-weak commutativity of single-valued self-maps due to Pant [16]. He has observed that the point-wise R-weak commutativity is more general than their compatibility. For details on compatibility of a hybrid pair, refer to [8, 9].

It appears that Ahmad and Imdad [1] have considered a hybrid pair T, F commuting in the sense FTx = TFx, and we shall follow the same notion throughout this paper. Following Jungck [8] and Jungck and Rhoades [9], Singh and Mishra [24] give the following definition.

Definition 2: Maps $T : K \to X$ and $F : K \to CL(X)$ are weakly compatible if they commute at their coincidence points, i.e., if TFx = FTx whenever $Tx \in Fx \subseteq K$

For an excellent discussion on the role of weak compatibility in fixed point considerations, one may refer to [9] when $T : X \to X$ and $F : X \to B(X)$, the set of all nonempty bounded subsets of X. We remark that

Commutativity $\Rightarrow$ Weak commutativity $\Rightarrow$ Compatibility $\Rightarrow$ Weak compatibility

However, the reverse implications are not true. Nevertheless, all these notions for T and F are equivalent at a coincidence point z (that is, when $Tz \in Fz$). Further, if T and F both are single-valued maps then weak compatibility of T and F is equivalent to R-weak commutativity of T and F at their coincidence points. Example 1 (below) shows that an R-weakly commuting hybrid pair T, F need not be weakly compatible. Indeed, R-weak commutativity of a hybrid pair of maps at coincidence points is more general than their weak compatibility. Following Itoh and Takahashi [7], Singh and Mishra [24] also give the following :

Definition 3 : Maps $T : K \to X$ and $F : K \to CL(X)$ are commuting at a point $x \in K$ if $TFx \subset FTx$ whenever $Tx \in Fx \subset K$ and $Tx \in K$. T and F are commuting on K if they are commuting at each point $x \in K$.

From now onward, the above commutativity will be called Itoh-Takahashi commutativity or simply (IT)-commutativity.

The example 1 of [24] shows that (IT)-commutativity of T and F at a coincidence points is indeed more general than their weak compatibility at the same point.

Remark 1 : In view of (1), (IT)-commutativity of T, F at a coincidence point z is equivalent to their R-weak commutativity at z.

On the other hand, we have introduced the notion of compatibility of type(N) in [19] for hybrid pair (T, F), where $T : X \to X$ and $F : X \to CB(X)$ and proved that notion of compatibility of type(N) is more general than weak compatibility of T and F. Further, in [21] we have proved that compatibility of type (N) is more general than commutativity of T and F at their coincidence point, while commutativity of T and F at their coincidence point is more general than weak compatibility of T and F proved by Singh and Mishra in [23]. Recently we have proved in [22] that compatibility of type (N) and F-weak commutativity introduced by Kamran [10] are equivalent at coincidence points. In the paper [20] we have extend the notion of compatibility of type (N) for non-self hybrid pair T and F and show that it is more general than (IT)-commutativity introduced by Singh and Mishra [24]. We thus have generalized the results of Singh and Mishra [24] and others.

Definition 4 : [20] Suppose $T : K \to X$ and $F : K \to CL(X)$. The pair (T, F) is called generalized compatible of type (N) iff $T(x) \in F(x)$, $F(x) \subset K$ implies that $TT(x) \in FT(x)$.

Note : If we put K = X in definition 5 we get our definition of compatibility of type (N) introduced in [19].

Lemma 1 : [20] (IT)-commutativity of the hybrid pair, (T, F) implies generalized compatibility of type (N) but not conversely.

Example 1 of [20] shows that converse is not true in general.

Definition 5 : Mappings $S : X \to CB(X)$ and $I : X \to X$ are called compatible if $Fx \in CB(X)$ for all $x \in X$ and $H(SIx_n, ISx_n) \to 0$, as $n \to \infty$ whenever (x_n) is a seq. x in X such that $Sx_n \to M \in CB(X)$ and $Ix_n \to l \in M$ as $n \to \infty$.

Following Singh and Mishra [24]. Pathak and Mishra [17] have introduced the notion of R-sequentially commuting mappings for a hybrid pair of single valued and multi-valued maps.

Definition 6 : [17] Let K be a nonempty subset of a metric space X and $I : K \to X$ and $S:K \to CL(X)$ be respectively single-valued and multi-valued mappings. Then I and S will be called R-sequentially commuting on K if for a given sequence $(x_n\} \subset K$ with $\lim Ix_n \in K$, there exists $R > 0$ such that

$$\lim_n D(Iy, SIx_n) \leq R \lim_n D(Ix_n, Sx_n) \tag{*}$$

for each $y \in K \cap \lim_n Sx_n$

If $x_n = x (x \in K)$ for all $n \in N$ (naturals), $Ix \in K$ and (*) holds for some $R > 0$, then I and S have been defined to be pointwise R-weakly commuting at $x \in K$ (see [Def. 1]). If it holds for all $x \in K$, then I and S are called R-weakly commuting on K. Further, if R = 1, they we get the definition of weak commutativity of I and S on K due to Hadzic and Gajec [4]. If Ix S: $X \to X$, then as mentioned in [17], we recover the definitions of pointwise R-weak conmmutativity and R-commutativity of single-valued self-maps due to Pant [16] and all the remarks as given in [17] apply.

Pathak and Mishra [17] have also introduced the following ;

Definition 7: Maps $I : K \to X$ and $S : K \to CL(X)$ are to be called sequentially commuting (or s-commuting) at a point $x \in K$ if

$$I(\lim_n Sx_n) \subset SIx \tag{**}$$

whenever there exists a sequence $\{x_n\} \subset K$ such that $\lim_n Ix_n = x \in \lim_n Sx_n \in CL(X)$.

Definition 8 : [17] If $x_n = x$ for all $n \in N$, in Definition 7, then the maps I and S will be said to be weakly s-commuting at a point $x \in K$.

Remark 2 : We remarked that to form $I(\lim Sx_n)$,IIx, SIx the restrictions, $\lim_n Sx_n \subseteq K$, $Ix \in K$ must be included in definition 7 and 8.

Remark 3 : With this restrictions weak s-commutativity (particular case of def. 7) defined by Pathak and Mishra [17] is equivalent to (IT) - commutativity.

The example 1 of [17] shows that s-commutativity of I and S is indeed more general than their R-sequential commutativity (and hence their pointwise R-commutativity and compatibility).

Definition 9: ([3]) Let R^+ denote the set of all non-negative real numbers, and let $A \subset R^+$. A function $\varphi : A \to R^+$ is upper semi continuous from the right if $\lim_{x \to u^+} \sup \varphi(x) \leq \varphi(u)$ for all $u \in A$.

A function $\varphi : R^+ \to R^+$ is said to satisfy (ϕ)-conditions if :

(i) φ is upper semi-continuous from the right on $(0, \infty)$ with $\varphi(t) < t$ for all $t > 0$, and

(ii) There exists a real number $s > 0$ such that φ is non-decreasing on (0, s] and

$\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t \in (0, s]$, where φ^n denotes the composition of φ with itself n

Times and $\varphi^0(t) = t$.

Let Γ denote the set of all functions which satisfy the (ϕ)-condition.

The following lemmas will be useful in proving our main results.

Lemma 2 : [17] Let (X, d) be a metric space and $I, J : X \to X$ and $S, T : X \to CL(X)$ be such that $S(X) \subset J(X)$ and $T(X) \subset I(X)$ and for all $x, y \in X$,

$$H(Sx, Ty) \leq \varphi(aL(x, y) + (1 - a) N(x, y)), \qquad (2)$$

where $a \in [0, 1]$, $\varphi : R^+ \to R^+$ is upper semi-continuous from the right on $(0, \infty)$ with $\varphi(t) < t$ for all $t > 0$, and

$L(x, y) = \max\{d(Ix, Jy), D(Ix, Sx), D(Jy, Ty), ½[D(Ix, Ty) + D(Jy, Sx)]\}$,

$N(x, y) = [\max\{d^2(Ix, Jy), D(Ix, Sx) D(Jy, Ty), D(Ix, Ty) D(Jy, Sx), ½ D(Ix, Sx) D(Jy, Sx), ½ D(Jy, Ty) D(Ix, Ty)\}]^{1/2}$

Then $\inf_{x\in X} D(Ix, Sx) = 0 = \inf_{x\in X} D(Jx, Tx)$

Lemma 3 [17] : Let X, I, J, S, T and φ be as defined Lemma 2 such that the inequality (2) holds. If $Ix \in Sx$ for some $x \in X$, then there exists a $y \in X$ such that $Ix = Jy$ and $Jy \in Ty$.

Lemma 4 ([18]) : Let $\varphi : R^+ \to R^+$ be a non-decreasing functions such that

(i) $\varphi(t+) < t$ for all $t > 0$ and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$.

Then there exists a strictly increasing function $\psi : R^+ \to R^+$ such that

(ii) $\varphi(t) < \psi(t)$ for all $t > 0$ and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$.

Lemma 5 ([3]) : If $\varphi \in \Gamma$, then there exists a function $\psi : R^+ \to R^+$ such that.

(i) ψ is upper semi-continuous from the right with $\varphi(t) \leq \psi(t) < t$ for all $t > 0$.

(ii) ψ is strictly increasing with $\varphi(t) < \psi(t)$ for $t \in (0, s]$, $s > 0$ and $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for $t \in (0, s]$.

The following theorem is the main result of Chang [3, Theorem 1].

Theorem A : Let (X, d) be a complete metric space, let I, J be two functions from X into X, and let $S, T : X \to CB(X)$ be two set-valued functions with $SX \subset JX$ and $TX \subset IX$. If there exists $\varphi \in \Gamma$ such that for all x, y in X,

$$H(Sx, Ty) \leq \varphi(\max\{d(Ix, Jy), D(Ix, Sx), D(Jy, Ty), ½D[(Ix, Ty) + D(Jy, Sx)]\}), \qquad (C)$$

then there exists a sequence $\{x_n\}$ in X such that $Ix_{2n} \to z$ and $Jx_{2n-1} \to z$ for some z in X and $D(Ix_{2n}, Sx_{2n}) \to 0$, $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$. Moreover, if $Iz = z$ and T and J are compatible, then $z \in Sz$ and $Jz \in Tz$. That is, z is a common fixed point of I and S, and z is a coincidence point of J and T.

The example 2 of [17] shows that Theorem A in its present form is incorrect.

Example 3 [17] : Let X = [0, 1] with absolute value metric d and let $\varphi : R^+ \to R^+$ be defined by $\varphi(t) = t^2$ for $t \in [0, 1)$ and $\varphi(t) = ½$ for $t \geq 1$. Define $I = J : X \to X$ and

$S = T : X \to CB(X)$ by $Ix = 1 - x$, $x \in X$ and $Sx = \{0, 1/3, 2/3, 1\}$ for all $x \in X$. Then for each x, $y \in X$ and $\varphi \in \Gamma$, we have

$$II(Sx, Ty) = 0 \leq \psi(\max\{d(Ix, Jy), D(Ix, Sx), D(Jy, Ty), ½D[(Ix, Ty) + D(Jy, Sx)]\}),$$

and for all sequence $\{x_n\} \subset X$ defined by $x_n = 1/n$ for all $n \in N$, we have Sx_n, $Tx_n \rightarrow \{0, 1/3, 2/3, 1\} = M$, $Ix_n, Jx_n = 1 - 1/n \rightarrow 1 \in M \subset X$, $D(Ix_{2n}, Sx_{2n}) \rightarrow 0$ and $D(Jx_{2n-1}, Tx_{2n-1}) \rightarrow 0$ as $n \rightarrow \infty$. Also, $z = 1/2 \in X$ is such that $Iz = z$ and for $\{x_n\}$ as defined above we have $\lim_n H(TJx_n, JTx_n) = 0$, that is, T and J are compatible. Thus, all the conditions of Theorem A are satisfied. Evidently $z \notin Sz$, $Jz \notin Tz$, that is, $z = 1/2$ is neither a common fixed point of I and S nor it is a coincidence point of J and T.

Following theorem will be useful for the proof our main theorem [17].

Theorem 1 : Let (X, d) be a complete metric space, and let $I, J : X \rightarrow X$, $S, T : X \rightarrow CL(X)$. Let A be a nonempty subset of X such that I(A) and J(A) are closed subsets of X, and $Tx \subseteq I(A)$ and $Sx \subseteq J(A)$ for all $x \in A$ and there exists a $\varphi \in \Gamma$ such that for all $x, y \in X$, (2) holds, Then

(i) $F = \{Ix : x \in X \text{ and } Ix \in Sx\} \neq \phi$,

(ii) $G = \{Jx : x \in X \text{ and } Jx \in Tx\} \neq \phi$,

(iii) F = G if A = X. and F = G is closed.

Theorem 2: Let (X, d) be a complete metric space, and let $I, J : X \rightarrow X$, $S, T : X \rightarrow CL(X)$ be such that $S(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$. If there exists a $\varphi \in \Gamma$ such that for all $x, y \in X$, (2) holds, then there is a sequence $\{x_n\}$in X such that $Ix_{2n} \rightarrow z$ and $Jx_{2n-1} \rightarrow z$ for some $z \in X$ and $D(Ix_{2n}, Sx_{2n}) \rightarrow 0$, $D(Jx_{2n-1}, Tx_{2n-1}) \rightarrow 0$ as $n \rightarrow \infty$. Moreover,

(i) if $Iz \in Sz$ and $d(Iz, z) \leq D(z, Sx)$ for all $x \in X$, then $z \in Sz$, and if $d(Iz, z) \leq D(z, Tx)$ for all $x \in X$, J and T are weakly s-commuting, then $Jz \in Tz$.

(ii) if $Iz \in Tz$ and $d(Jz, z) \leq D(z, Tx)$ for all $x \in X$. then $z \in Tz$; and if $d(Jz, z) \leq D(z, Sx)$ for all $x \in X$, I and S are weakly s-commuting, then $Iz \in Sz$.

(iii) if $Iz = z$ and J and T are weakly s-commuting, then $z \in Sz$ and $Jz \in Tz$.

(iv) if $Jz = z$ and I and S are weakly s-commuting, then $z \in Tz$ and $Iz \in Sz$.

Remark 4 : Theorem 1 of Naidu [15] and Theorem 9 of Sastry, Naidu and Prasad [18] follow as direct corollaries of Theorem 1.

Remark 5 : For a = 1, Example 10 of Sastry, Naidu and Prasad [18] shows that Theorem 1 fails if $½[D(Ix, Ty) + D(Jy, Sx)]$ is replaced by max $\{D(Ix, Ty), D(Jy, Sx)\}$ even if S = T, $I = J = i_d$ (the identity mapping on X) and φ is continuous on R^+.

Remark 6 : If (1) is assumed to be valid only for those $x, y \in X$ for which $Ix \neq Jy$,

$Ix \notin Sx$ and $Jy \notin Ty$ instead of all $x, y \in X$, then we conclude from Theorem I that: either $F = \{Ix : x \in X \text{ and } Ix \in Sx\} \neq \phi$ or $G = \{Jx : x \in X \text{ and } Jx \in Tx\} \neq \phi$.

3. MAIN RESULTS

We introduce:

Let (X, d) be a metric space, $K \subseteq X$, $I : K \rightarrow X$. $T : K \rightarrow CB(X)$ or CL(X) as the case may be.

Definition: Mapping $I : K \rightarrow X$ is said to be sequentially T-weak commuting at $x \in K$. if

$I \lim Ix_n \in TIx$ whenever there exists a sequence $\{x_n\} \subseteq K$ such that

$\lim Ix_n = x \in K$, and $\lim Tx_n \subset K$.

If $x_n = x$ for all $n \in N$, then we have,

Definition: $I : K \rightarrow X$ is said to be generalized T-weakly commuting at $x \in K$ if

$IIx \in TIx$ whenever $Ix \in K$, $Tx \subseteq K$.

If K = X, we have following def. due to Kamran [10]

Definition : $I : X \rightarrow X$ is said to T-weak commuting at x if $IIx \in TIx$.

Lemma : Sequential commutativity implies sequential T-weak commutativity but not conversely.

Proof : Suppose I and T are sequentially commuting mapping, then we have,

$I(\lim Tx_n) \subseteq TIx$ and there exists a sequence $\{x_n\} \subseteq K$ such that

$\lim_n Ix_n = x \in \lim_n Tx_n$, and $\lim_n Tx_n \subseteq K$.

Now $\lim_n Ix_n \in \lim Tx_n$ gives $I(\lim Ix_n) \in I(\lim Tx_n)$ and since $I(\lim_n Tx_n) \subseteq TIx$, we have $I(\lim_n Ix_n) \in TIx_n$ and Hence I is sequentially T-weak commuting.

To prove the converse we provide following example.

Example 4 : Let $X = [0, \infty)$, $K = [0, 1]$ Let $T : K \to X$ be defined by

$T(x) = 1 - x$ if $x \in [0, 1]$ and

$$F(x) = \begin{cases} [\frac{1}{2}, 1] & \text{if } x \in [0, \frac{1}{2}] \\ [0, \frac{1}{2}] & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Take any sequence x_n in in $K = [0, 1]$. We consider two cases,

(i) $\{x_n\} \subseteq [0, ½]$ (ii) $\{x_n\} \in (½, 1]$

Case I : First suppose

$x_n \in [0, ½]$. Then $Ix_n = 1 - x_n \in [½, 1]$ and $\lim_n Ix_n$

$= x$ (say) $\in [½, 1] = \lim_n Fx_n$. Now $T(\lim_n Fx_n) = T[½, 1]$

$= [0, ½]$ and $FTx = F(1-x) = [½, 1]$ since $x \in [½, 1]$ then $1 - x \in [0, ½]$. Clearly

$T(\lim_n Fx_n) = [0, ½] \not\subset FTx = [½, 1]$ for all $x_n \in [0, ½]$, $x \in [½, 1]$

Case II : Suppose $\{x_n\} \subseteq (½, 1]$. Then $Tx_n \in [0, ½]$ and also $\lim_n Tx_n = x \in [0, 1/2]$.

Now $Fx_n = [0, ½]$ and $\lim_n Fx_n = [0, ½]$ therefore $\lim_n Tx_n \in \lim_n Fx_n$.

But $T(\lim_n Fx_n) = T[0, ½] = [½, 1]$ and

$FTx = F(1-x) = [0, ½]$ since $1-x \in (½, 1]$. Obviously $T(\lim_n Fx_n) \not\subset FTx$ for any sequence $x_n \in [½, 1]$.

Therefore T and F are not s-commuting. However, taking the sequence $x_n = ½ + 1/_{2}{}^{n+1}$, $n \in N$, We have $\lim Tx_n = ½$ and $T(\lim Tx_n) = ½$. Further, $FTx = FT(½) = F(½) = [½, 1]$ and Hence $T(\lim Tx_n) = ½ \in FT(½) = [½, 1]$.

Therefore T and F are sequentially T-weakly and generalized T-weakly commuting at $x = ½ \in [0, 1]$

Theorem 3 : Let (X, d) be a complete metric space and K be a subset of X. Let $I, J : K \to X$. $S, T: X \to CL(X)$ be such that $S(K) \subseteq J(K)$ and $T(K) \subseteq I(K)$. If there exists a $\phi \in \Phi$ such that for all $x, y \in K$,

$H(Sx, Ty) \leq \phi(aL(x, y) + (1 - a)N(x, y)$, where $a \in [0, 1]$ $\phi : R^+ \to R^+$ is upper semi-continuous from the right on $(0, \infty)$ with $\phi(t) < t$ for all $t > 0$ and

$L(x, y) = \max \{d(Tx, Jy), D(Ix, Sx), D(Jy, Ty), ½[D(Ix, Ty) + D(Jy, Sx)]\}$

$N(x, y) = [\max \{d^2(Ix, Jy), D(Ix, Sx) D(Jy, Ty), D(Ix, Ty), D(Jy, Sx),$

$½D(Ix, Sx) D(Jy, Sx), ½D(Jy, Ty) D(Ix, Iy)\}]^{1/2}$ (3)

Holds, then there is a sequence $\{x_n\}$ in X such that $Ix_n \to z$, $Jx_{2n+1} \to z$ for same $z \in X$ and $D(Ix_{2n}, Sx_{2n}) \to 0$, $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$ Moreover,

(i) If $Iz \in Sz$ and $d(Iz, z) \le D(z, Sx)$ for all $x \in K$, then $z \in Sz$, and if $d(Iz, z) \le d(z, Tx)$ for all $x \in K$. J and T are generalized T-weak commuting on $G = \{x : Jx \in Tx\}$, then $Jz \in Tz$.

(ii) If $Jz \in Tz$ and $d(Jz, z) \le D(z, Tx)$ for all $x \in K$. then $z \in Tz$; and If $d(Jz, z) \le D(z, Sx)$ for all $x \in K$. and I and S are generalized T-weak commuting on $F = \{x : Ix \in Sx\}$, then $Iz \in Sz$.

(iii) If $Iz = z$ and J and T are generalized T-weak commuting on $G = \{x : Jx \in Tx\}$, then $z \in Sz$ and $Jz \in Tz$.

(iv) If $Jz = z$ and I and S are generalized T-weak commuting on $F = \{x : Ix \in Sx\}$, then $z \in Tz$ and $Iz \in Sz$.

Proof : Although the first part of the proof is same as given by Pathak and Mishra [17], Yet we provide it for completeness.

We can construct a sequence $\{x_n\}^{\infty}_{n=0} \subset K$ such that $Jx_{2n+1} \in Sx_{2n}$, $Ix_{2n+2} \in Tx_{2n+1}$ (n = 0, 1, 2 .....) and the sequence $\{Ix_{2n}\}$, $\{Jx_{2n-1}\}$ are Cauchy sequences which converge to the same limit $z \in X$ and $D(Ix_{2n}, Sx_{2n}) \to 0$, $D(Jx_{2n-1}, Tx_{2n-1}) \to 0$ as $n \to \infty$. It then follows that $D(z, Sx_{2n}) \to 0$ and $D(z, Tx_{2n-1}) \to 0$ as $n \to \infty$.

(i) Suppose $Iz \in Sz$, $d(Iz, z) \le D(z, Sz)$ and J and T are generalized T-weak commuting on $\{x : Jx \in Tx\}$.

Choose $m \in N$ such that

$\sup\{d(Ix_{2n}, z), d(Jx_{2n-1}, z), D(z, Sx_{2n}), D(z, Tx_{2n-1}) : n \ge m\} < s$.

Then for $n \ge m$ we have

$$\begin{aligned} D(z, Sz) &\le d(z, Ix_{2n}) + D(Ix_{2n}, Sz) \\ &\le d(z, Ix_{2n}) + H(Sz, Tx_{2n-1}) \\ &\le d(z, Ix_{2n}) + \varphi(aL(z, x_{2n-1}) + (1 - a)\, N(z, x_{2n-1})), \quad (4) \end{aligned}$$

where

$$\begin{aligned} L(z, x_{2n-1}) &= \max \{d(Iz, Jx_{2n-1}), D(Jz, Sz), D(Jx_{2n-1}, Tx_{2n-1}), \\ &\quad \tfrac{1}{2}[D(Iz, Tx_{2n-1}) + D(Jx_{2n-1}, Sz)]\} \\ &\le \max \{d(Iz, Jx_{2n-1}), 0, d(x_{2n-1}, Tx_{2n-1}) \\ &\quad \tfrac{1}{2}[d(Jz, z) + D(z, Tx_{2n-1}), d(Jx_{2n-1}, z)\, D(z, Tx_{2n-1})]\} \\ &\to \max \{d(Iz, z), 0, 0, \tfrac{1}{2}d(Iz, z)\} \text{ as } n \to \infty \end{aligned}$$

i.e.

$\lim_n L(z, x_{2n-1}) \le D(z, Sz)$;

and $N(z, x_{2n-1}) \le [\max\{d^2(Iz, z), 0, 0, 0, 0\}]^{1/2}$ as $n \to \infty$

i.e. $\lim_n N(z, x_{2n-1}) \le D(z, Sz)$

Hence making $n \to \infty$ in (4), we obtain,

$D(z, Sz) \le 0 + \varphi(aD(z, Sz) + (1 - a)\, D(z, Sz))$,

that is, $D(z, Sz) = 0$ and so $z \in Sz$. Choose $z' \in X$ such that $Jz' = z$, then

$D(z, Tz') \le H(Sz, Tz')$ (5)

$\le \varphi(aL(z, z') + (1 - a)\, N(z, z'))$ where

$L(z, z') = \max \{d(Iz, Jz'), D(Iz, Sz), D(Jz', Tz'), \tfrac{1}{2}[D(Iz, Tz') + D(Jz', Sz)]\}$

$\leq \max\{d(Iz, z), D(Iz, Sz), D(z, Tz'), ½[d(Iz, z) + D(z, Tz') + D(z, Sz)]\}$

$= maz\{d(Iz, z), D(z, Tz')\} \leq D(z, Tz')$

And $N(z, z') \leq [\max\{d^2(Iz, z), 0, 0, 0, ½D(z, Tz') [d(Iz, z) + d(z, Tz')]\}]^{1/2} \leq D(z, Tz')$

Hence by (5) $D(z, Tz') \leq \varphi (D(z, Tz'))$, and so $D(z, Tz') = 0$; i.e., $Jz' = z \in Tz'$

Since J and T are generalized T-weak commuting at z', $Jz' \in Tz'$, we have

$JJz' \in TJz'$ which implies that $Jz \in Tz$.

(ii) The proof is analogous to the proof of (i) due to symmetry.

(iii) Suppose $Iz = z$ and J and T are generalized T-weak commuting on $\{x: Jx \in Tx\}$. Choose m as in (i), then for all $n \geq m$

$D(z, Sz) \leq d(z, Ix_{2n}) + D(Ix_{2n}, Sz)$ (6)

$\leq d(z, Ix_{2n}) + H(Sz, Tx_{2n-1})$

$\leq d(z, Ix_{2n}) + \varphi(aL(z, x_{2n-1}) + (1 – a) N(z, x_{2n-1})$, where

$L(z, x_{2n-1}) \to \max\{0, D(z, Sz), 0, ½D(z, Sz)\}$ as $n \to \infty$,

i.e. $\lim_n L(z, x_{2n-1}) = D(z, Sz)$

and $N(z, x_{2n-1}) \to [\max \{0, 0, 0, ½D^2(z, Sz), 0\}]^{1/2}$ as $n \to \infty$,

i.e. $\lim_n N(z, x_{2n-1}) = D(z, Sz)$.

Making $n \to \infty$ in (6), we obtain

$D(z, Sz) \leq 0 + \varphi(aD(z, Sz) + [(1 – 1) / \sqrt{2}]D(z, Sz) \leq D(z, Sz)$,

which implies $D(z, Sz) = 0$ and so $z \in Sz$. Choose $z' \in X$ such that $Jz' = z$, then

$D(z, Tz') \leq H(Sz, Tz') \leq \varphi (aL(z, z') + (1 – a)N(z, z'))$ where

$L(z, z') = \max\{d(Iz, Jz'), D(Iz, Sz), D(Jz', Tz'), ½[D(Iz, Tz') + D(Jz, Sz)]\} = D(z, Tz')$

and

$N(z, z') = [\max\{d^2(Iz, Jz'), D(Iz, Sz) D(Jz', Tz'), D(Iz, Tz') D(Jz', Sz), ½D(Iz, Sz) D(Jz', Sz), ½D(Jz', Tz') D(Iz, Tz')\}]^{1/2} = (1/\sqrt{2})D(z, Tz')$

Hence

$D(z, Tz') \leq \varphi (aD(z, Tz') + [(1 – a) /\sqrt{2}] D(z, Tz') \leq D(z, Tz')$

It follows that $D(z, Tz') = 0$ and so $Jz' = z \in Tz'$. Since J and T are generalized T-weak commuting at z', $Jz' \in Tz'$, we have $JJz' \in JTz'$. Hence $Jz \in Tz$.

(iv)Due to symmetry, the proof is analogous to the proof of (iii).

Remark 7: As it has been shown in [22] that the concepts compatibility of type (N) and T-weak commutativity are equivalent at coincidence points, the phrase "I and S (resp. J and T) are generalized T-weak commuting on $\{x :Ix \in Sx\}$, (resp. $\{x : Jx \in Tx\}$)" in every part of the theorem 3 may be elegantly replaced by the phrase "I and S(resp. J and T) are compatible of type (N)".

4. Conclusion

Taking K= X and replacing the condition of generalized T-weak

commutativity by weak s-commutativity from theorem 3, we get theorem 2. It is also notable that generalized T-weak commutativity is required only on coincidence points of the mappings while in result of Pathak and Mishra [17] weak s-commutativity is required on all of X. Further, in remark 2 we have suggested two essential restrictions

To impose for defining weakly s-commuting mappings i.e. to define weakly s-commuting mappings we have to form $I(\lim_n Sx_n)$, SIx & I(Ix) and therefore the restrictions $\lim_n Sx_n \subseteq K$ & $Ix \in K$ must be included in the definitions 7 and definition 8,with these restrictions, weak s-commutativity defined by Pathak and Mishra [16] is nothing but (IT)-commutativity defined by Singh and Mishra[24].Finally we claim following obvious lemma ,

Lemma7: The assumption of generalized T-weak commutativity at coincidence points (generalized compatibility of type (N)) in case of hybrid pair (I, S) (resp. (J, T)) is the minimal condition for existence of common fixed point of the hybrid pairs.

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