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An Unsteady Viscous Incompressible Flow In A Porous Medium Between Two Impermeable Parallel Plates Impulsively Stopped From Relatively Motion

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Abstract: *The problem presented in this paper is an unsteady flow of viscous incompressible homogeneous fluid through a porous slab bounded between two impermeable parallel plates separated by a finite gap with the neglect pressure gradient. The flow is generated by the motion of one of the plates with a constant velocity parallel to itself while the other is kept rest. When the steady state is reached the moving plate is suddenly stopped and the subsequent motion investigated employing the Laplace Transform technique to obtain the fluid-velocity field. Expressions for ensuing flow-rate and skin friction on the boundaries have been obtained.*

Variations of these flow parameters with time and the porosity coefficient have been illustrated based on which some conclusions have been reached using Matlab

1. INTRODUCTION

The viscous flow between two parallel plates is a classical problem, we know that the fluid exerts viscosity effect when there is a tendency of shear flow of the fluid. Fluid flow through porous media have been attracting the attention of Engineers and Applied Mathematicians for the last one and half century as it has various applications in certain devices such as electrostatic precipitation, magneto-hydrodynamic (MHD)power generators, MHD pumps, accelerators, petroleum industry, polymer technology, aerodynamics heating, purification of molten metals from non-metallic inclusions and fluid droplets-sprays.

Impulsive flows of incompressible fluids between rigid boundaries is attracting the attention of researchers due to its wide range of application in several branches of sciences and technology. Knowledge of the flow through porous media is immensely useful in the efficient recovery of crude oil from the porous reservoir rocks by the displacement with immissible water (Rudraiah.et.al [1]). Run-up and spin-up flows belong to this category of flow. Kazakai and Rivlin[2] investigated run-up and spin-up flows of non-Newtonian liquids in parallel plate and circular geometries. The flow field is arising from super position of waves propagated into the fluid and reflected back and forth at the boundaries has been investigated via the solution of initial value problem, later by Pattabhi Ramacharyulu and Appalaraju [3], Appala Raju.K [4] for flows through porous media and Ramakrishna [5] for second order liquids examined.T.Gnana Prasuna ,Pattabhi Ramacharyulu and Ramanamurthi [6] and T.Gnana Prasuna [7] investigated a run-up flow of a visco elastic fluid through a porous medium in parallel plate geometry. Run up flows were studied earlier by Ramakrishna[5] for fluids with particle suspension and Ramanamurthi [8] for second order fluid. Forced and natural flows were discussed by Schlitchting [9],Eckert and Drake [10] and Bansal [11].Pop and Soundalgekar [12] investigated free convection flow past an accelerated vertical infinite plate. The present authors[13] recently investigated an impulsive flow in porous slab bounded between two impermeable parallel plates

Here we investigate a special run-up retarding flow of a viscous incompressible homogeneous fluid through a porous slab of finite thickness bounded between two impermeable parallel plates.

Following Yamamoto and Yashida Z. [14], we adopt the following momentum equation for the flow through porous media which takes care of the fluid inertia and the viscous stress in addition to the classical Darcy's friction

$$\rho\left[\frac{\partial\bar{V}}{\partial t} + (\nabla.\bar{V})\nabla\right] = -\nabla P + \mu\nabla^2\bar{V} - \frac{\mu}{k}\bar{V}$$

where ρ is the fluid density

μ is the coefficient of viscosity

k is the Darcy coefficient of porosity of the medium

p is the pressure and

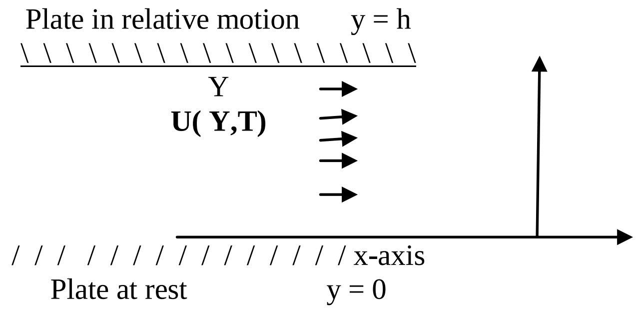
$\bar{V}$ is the fluid velocity vector

Initially, the flow is generated by a motion of one of the plates with a constant velocity. When the steady state is reached, the moving plate is suddenly stopped and the subsequent fluid-flow is examined.

Analytic expressions for the Velocity, Flow rate and Skin-Friction on the two plates are obtained, by employing Laplace Transformation technique and variation of all the flow parameters are shown with illustration and conclusions are drawn.

2. MATHEMATICAL FORMULATION

Consider a Cartesian frame of reference O(x,y,z) with origin is on a fixed plate. x-axis parallel to the moving plate and y-axisperpendicular to the plates



In this Cartesian frame, the fluid velocity can be taken as $\bar{V}$ = [u(y,t),o,o]

This evidently satisfied by the continuity equation

$$\nabla.\bar{V} = 0 \tag{1}$$

The momentum equation is in X-direction characterizing the ***flow through porous media*** is

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\mu}{k_1}u \tag{2}$$

Where ρ is the fluid density, μ is the coeff of viscosity and

k_1 is the Darcy coefficient of porosity of the medium

The present problem is solved in two stages. In the first stage the flow is generated by the motion of one of the plates with the constant velocity parallel to itself while the other is kept at rest and the second stage is concerns with the subsequent unsteady receding flow, when the moving plate (cited in stage-1) is stopped.

Stage – 1:

This is the steady state of the fluid-flow generated by the motion of the plate y = h with a velocity u_0 .

Let $u_s(y)$ be the velocity of the fluid.

The momentum equation in this case is

$0 = \frac{d^2u_s}{dy^2} - \frac{u_s}{k_1}$ (3)

With the boundary conditions
$u_s(\, y = 0) = 0 \,\&\, u_s(\, y = h) = u_0$ (4)

The solution of the equation (3) subject to the boundary condition (4) is

$u_s(y) = \frac{sinh\frac{y}{\sqrt{k_1}}}{sinh\frac{h}{\sqrt{k_1}}} \;=\; f(y) \;\; say$ (5)

Stage – 2:

After attaining the steady flow considered in stage – 1, the moving plate y = h is suddenly stopped.

Let $u^*(\mathrm{y,t})$ be the fluid velocity in the subsequent motion be given by

This satisfies the momentum equation

$\frac{\partial u^*}{\partial t} = \gamma \frac{\partial^2 u^*}{\partial y^2} - \gamma \frac{u^*}{k_1}$ (6)
Where $\gamma = \frac{\mu}{\rho}$ (7)
with the boundary conditions:
$u^*(0, \mathrm{t}) = 0$ and $u^*(\mathrm{h,t}) = 0$ (8)
and with the initial conditions:
$u^*(\mathrm{y}, 0) = \mathrm{f(y)}$ (9)
where f(y) given in equ(5))

The following system of non-dimensional quantities is introduced for the simplicity of presentation of the result

Y = hY; $\mathrm{t} = \frac{h^2T}{\gamma}$; $k_1 = Kh^2$; $u^*(\mathrm{y,t}) = \frac{\gamma}{h^2}\,\mathrm{u(Y,T)}$ (10)
where u_0 is the velocity of the moving plate which is generating the motion in the stage - 1

The momentum equation expressed in the non-dimensional form is

$\frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial Y^2} - D^2 u$ (11)

Where $D^2 = \frac{1}{K}$ (12)

and with the boundary condition:

u(o,T) = o & u(1,T) = o (13)

and the initial condition:

u(y,o) = $\frac{\sinh{(DY)}}{\sinh{(D)}}$ (14)

Let Laplace Transform of u(Y,T) be

$\bar{u}\,(Y,s) = \int_0^\infty u(Y,T)e^{-sT}dT$ (15)
Taking L.T. of the equation (11), we get

$\frac{d^2\bar{u}}{dY^2} - m^2\bar{u} = \frac{\sinh{(DY)}}{\sinh{(D)}}$ (16)

Where m = $\sqrt{s + D^2}$ (17)

With the boundary conditions are

$\bar{u}(0,s) = 0 = \bar{u}(1,s)$ (18)

The solution of (16) satisfy the boundary condition (18) is

$$\bar{u}\,(Y,s) = \frac{1}{s}\left(\frac{\sinh\,(DY)}{\sinh\,(D)} - \frac{\sinh\,(mY)}{\sinh\,(m)}\right) \tag{19}$$

The inverse Laplace transform of which is

$$u(Y,T) = L^{-1}\bar{u}\,(Y,s)$$

$$= L^{-1}\left[\frac{1}{s}\left(\frac{\sinh\,(DY)}{\sinh\,(D)} - \frac{\sinh\,(mY)}{\sinh\,(m)}\right)\right]$$

$$= \frac{\sinh(DY)}{\sinh(D)}H(t) - L^{-1}\left[\frac{\sinh\,(mY)}{s(\sinh(m))}\right] \tag{20}$$

Where

$H(t) = \begin{cases} o & t < 0 \\ 1 & t > 1 \end{cases}$ Is the Heaviside unit step functions

The Laplace inverse in the second term of (20) is obtained by using the Residue theorem (vide: Appendix)

The flow variables at this stage are

Velocity:

$$u(Y,T) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, n\, e^{-(n^2\pi^2+D^2)t}\sin\,(n\pi y)}{n^2\pi^2+D^2} \tag{21}$$

Flow-Rate:

The flow rate Q(t) is given by $Q = \int_0^1 u(y,s)\ dy$ (22)

the Laplace Transform of which is

$$\bar{Q} = \int_0^1 \bar{u}(y,s)\ dy$$

$$= \frac{\cosh(D)-1}{sD(\sinh(D))} - \frac{coshm-1}{m\, s\, sinhm} \tag{23}$$

Taking the inverse Laplace transform we get

Q(t) = $L^{-1}[\bar{Q}]$

$$= \sum_{n=1}^{\infty}(-1)^{2n-1}\ \frac{e^{-(n^2\pi^2+D^2)t}}{n^2\pi^2+D^2}$$

Shear-Stress:

The shear stress $\tau(y,t)$ is in the flow-region is

$$\tau = \mu\frac{du}{dy} \tag{24}$$

the Laplace transfrom of which is

$$\bar{\tau} = \mu\frac{d\bar{u}}{dy}$$

$$= \frac{\mu\, D\, coshDy}{s\, sinhD} - \frac{\mu\, m\, coshmy}{s\, sinhm} \tag{25}$$

Taking the inverse L.T. we get

$$\tau = 2\mu\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n^2\pi^2 e^{-(n^2\pi^2+D^2)t}}{n^2\pi^2+D^2}\cos\,(n\pi y) \tag{26}$$

Therefore the shear stress on the bottom plate (y = 0) is

$$\tau_{at\ y=0} = 2\mu\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n^2\pi^2 e^{-(n^2\pi^2+D^2)t}}{n^2\pi^2+D^2} \tag{27}$$

And that on the other plate (y = 1) is

$$\tau_{at\ y=1} = 2\mu \sum_{n=0}^{\infty}(-1)^{2n+1} \frac{n^2\pi^2 e^{-(n^2\pi^2+D^2)t}}{n^2\pi^2+D^2} \tag{28}$$

3. CONCLUSION

Numerical computations have been carried out for the values of different flow variables with diverse values of coefficient of porosity D at different time instants. It is noticed that the effect of porosity on the velocity profiles is to flatten type with the velocity attaining maximum near the middle of the plates and due to friction, it decrease towards the plates. This effect will be more predominant profiles and as D increases. Further the point of maximum velocity is shifted towards the upper plate (y=1).

(Vide Figs (1-10))

Fig 12 and Fig 13 illustrate the variation of the shear stress on the lower plate y=0 and upper plate y=1

Fig 13 shows the variation of the flow rate with D and t. $|Q| \to 0$ as D increases and also as t increases.

APPENDIX

Let $L\ f(t) = f(s)$ then

$$f(t) = L^{-1}\ f(s)$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f(s)\frac{\delta y}{\delta x} e^{st} ds$$

where γ is a vertical contour in the complex plane chosen so that all the poles of the integrand are to the left of it [15] Eric W. Weirstein, CRC, Concise Encyclopedia of Mathematics, 1999, page 176, [16] Sneddon I. N. , Use of Integral transforms, Tata Mac Graw-Hill, 1979.

Therefore $L^{-1}\ f(s)$ = Sum of the residues of all the poles of f(s) e^{st}

i.e. of f(s) inside the Bronwich Contour.

from (20)

$$u(Y,T) = \frac{\sinh(DY)}{\sinh(D)} H(t) - L^{-1}\left[\frac{\sinh(mY)}{s(\sinh(m))}\right]$$

The poles of the second term are

s = 0 and sinh(m) = 0

$$\Rightarrow m_n = n\pi i$$

$$s_n = -(n^2\pi^2 + D^2) \qquad n=0, 1, 2, 3\ldots$$

Residue at s=0:

$$R_1 = \frac{\sinh(DY)}{\sinh(D)}$$

Residue at s = s_n:

$$R_2 = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n\, n\, e^{-(n^2\pi^2+D^2)t} \sin(n\pi y)}{n^2\pi^2 + D^2}$$

Thus,

$$u(Y,T) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, n\, e^{-(n^2\pi^2+D^2)t} \sin(n\pi y)}{n^2\pi^2+D^2}$$

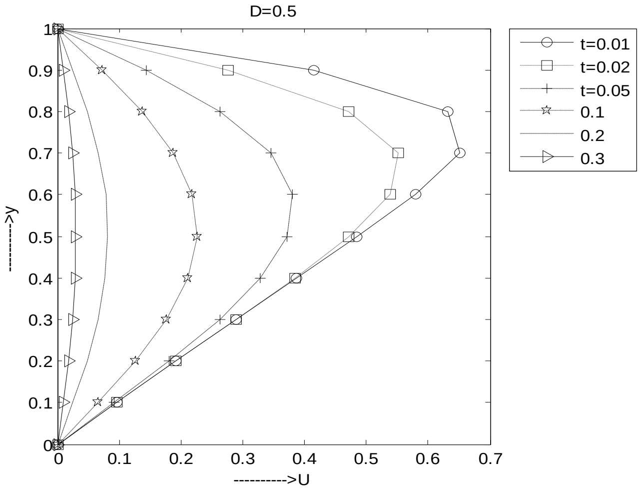


Fig.1. *Velocity profiles at different time instants when D=0.5*

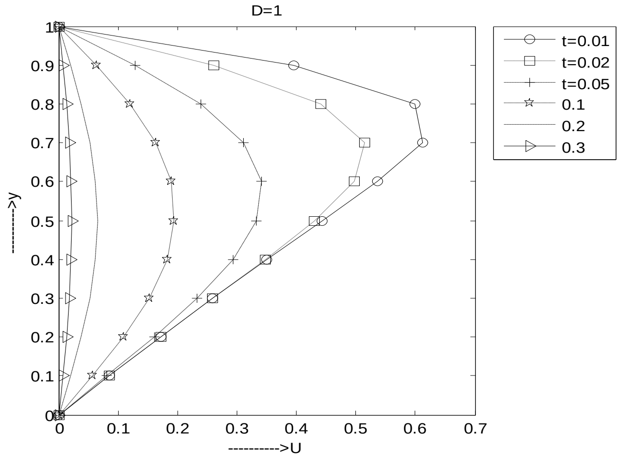


Fig.2. *Velocity profiles at different time instants when D=1*

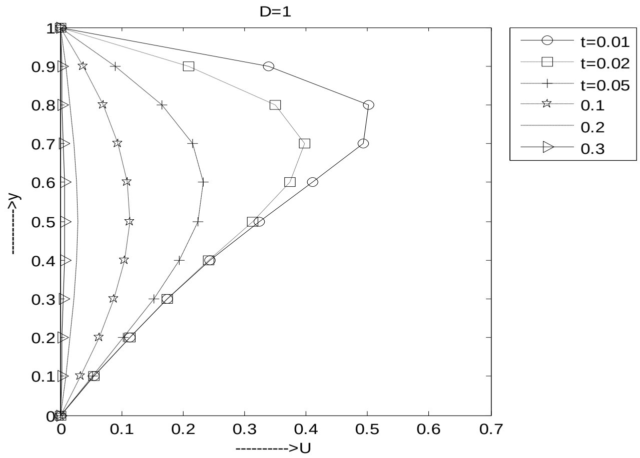


Fig.3. *Velocity profiles at different time instants when D=2*

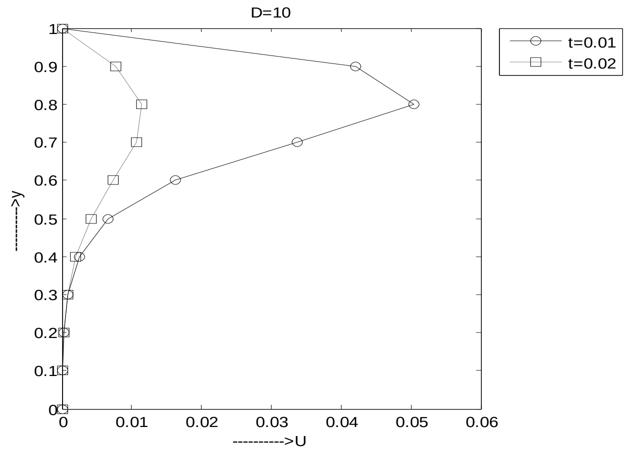


Fig4. *Velocity profiles at different time instants when D=10*

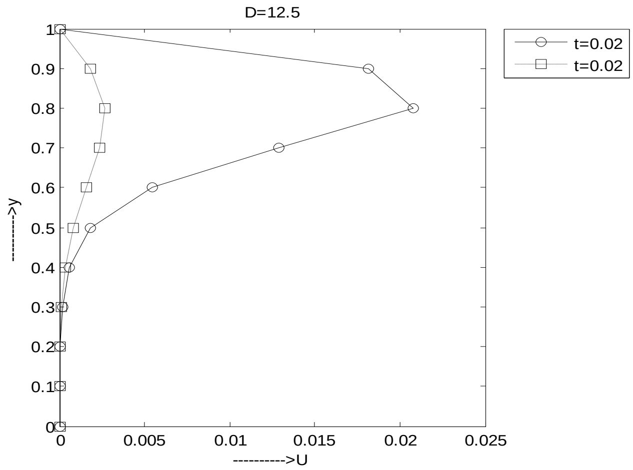


Fig.5. *Velocity profiles at different time instants when D=12.5*

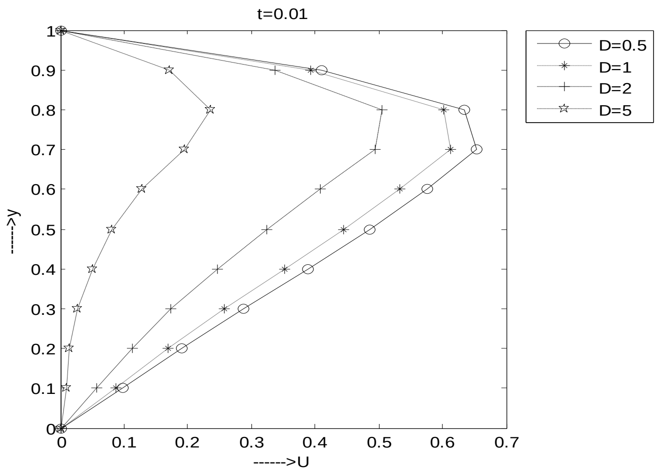


Fig.6. *Velocity profiles verses D at t = 0.01*

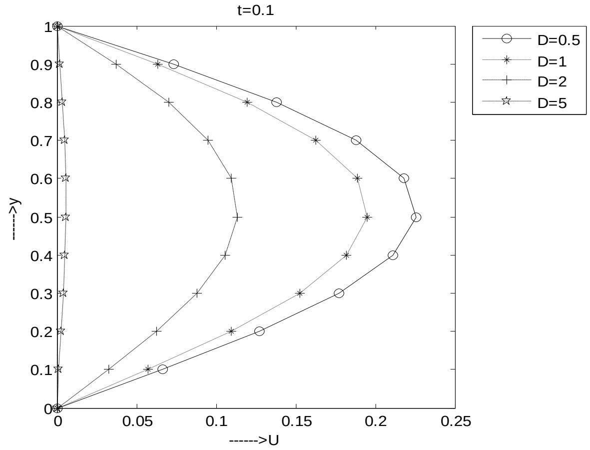


Fig.7. *Velocity profiles verses D at t = 0.1*

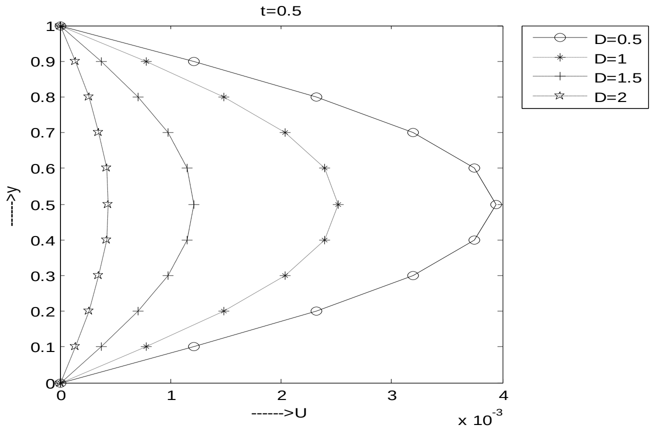


Fig.8. *Velocity profiles verses D at t = 0.5*

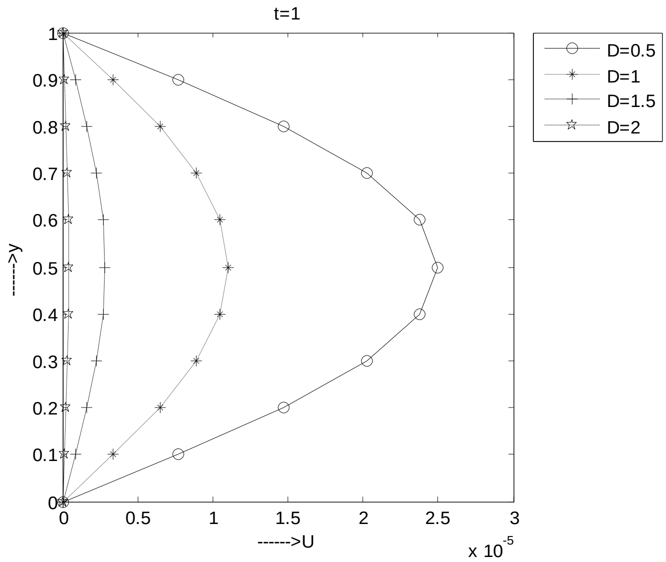


Fig.9. *Velocity profiles verses D at t = 1*

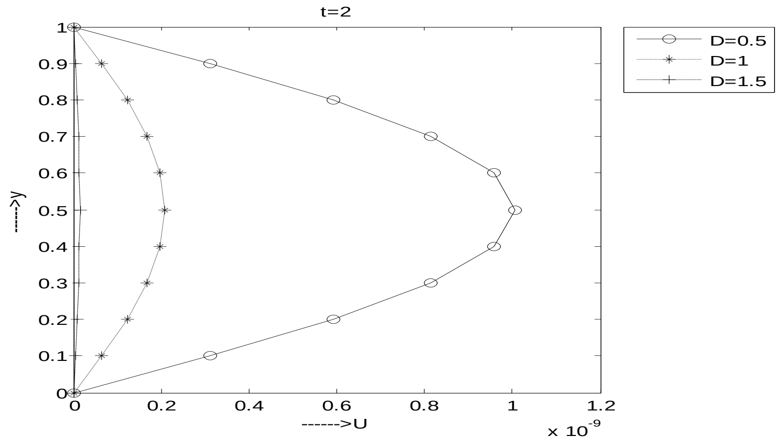


Fig.10. *Velocity profiles verses D at t = 2*

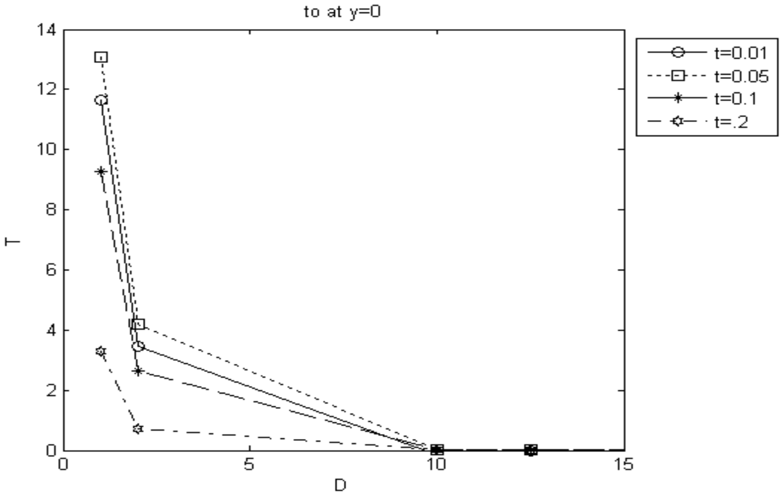


Fig.11. *Shear stress on the lower plate verses D at different t*

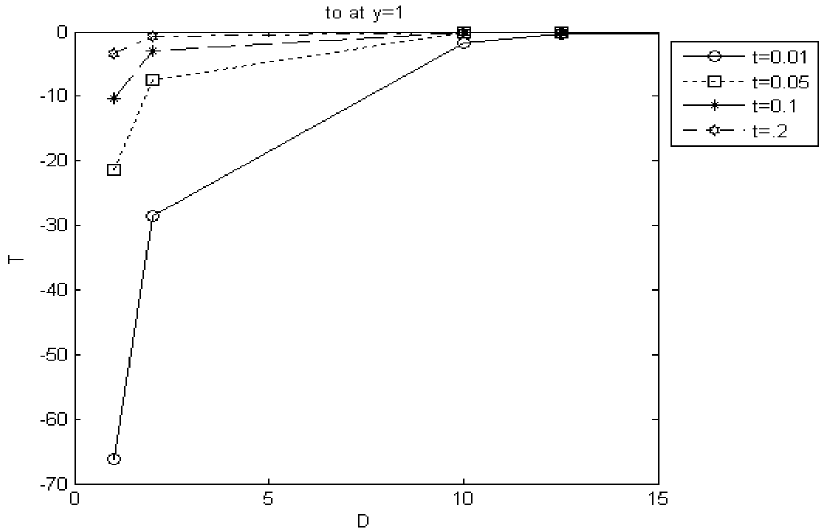


Fig.12. *Shear stress on the upper plate verses D at different t*

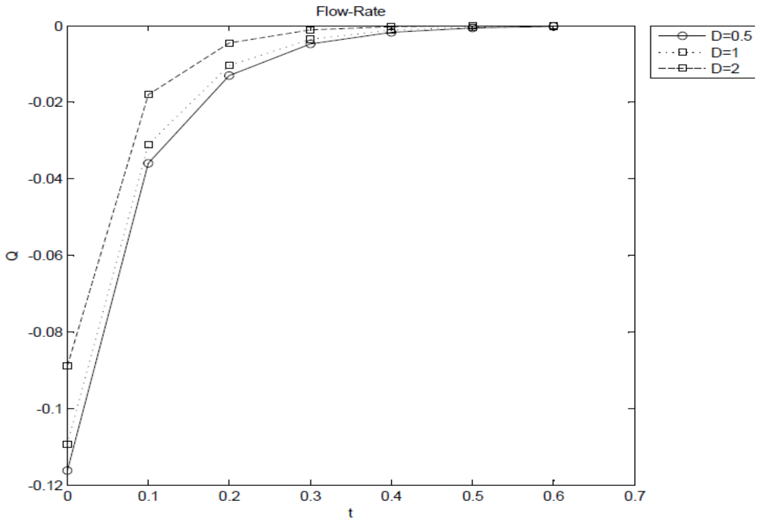


Fig 13. *Flow rate verses t for different values of D*

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