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Trignometric Inequations and Fuzzy Information Theory

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Abstract: *Some new trigonometric measures of fuzzy entropy involving trigonometric functions have been provided and their validity is checked by studying their essential properties and certain inequalities involving trigonometric angles of a convex polygon of n sides have been proved by making use of some concepts from fuzzy information theory.*

Key words: *Trigonometry, fuzzy information theory, entropy, fuzzy entropy trigonometric inequalities*

1. Introduction

For the probability distribution P = (p_1 , p_2 , ..., p_n), Shannon [8] obtained the measure of entropy as :

$$S_1(\text{P}) = -\sum_{i=1}^{n} p_i \log p_i \tag{1.1}$$

Which is a concave function and has maximum value when $p_1 = p_2 = ... = p_n = \frac{1}{n}$

Corresponding to (1.1), Deluca and Termini [2] suggested the measure of fuzzy entropy as

$$H_1(\text{A}) = -\sum_{i=1}^{n} \left[\mu_A(x_i) \log \mu_A(x_i) + (1-\mu_A(x_i)) \log \ 1-\mu_A(x_i) \right] \tag{1.2}$$

Which is a concave function and has maximum value when A is most fuzzy set.

Renyi's [7] probabilistic measure of entropy is given by

$$\text{H}^{\alpha}(\text{P}) = \frac{1}{1-\alpha} \log \sum_{i=1}^{n} p_i^{\alpha}, \quad \alpha > 0, \ \alpha \neq 1 \tag{1.3}$$

Corresponding to (1.3), Bhandari and Pal [1] suggested measure of fuzzy entropy as

$$\text{H}^{\alpha}(\text{A}) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \log \left[\mu_A^{\alpha}(x_i) + \ 1-\mu_A(x_i) \ ^{\alpha} \right]; \quad \alpha > 0, \ \alpha \neq 1 \tag{1.4}$$

Corresponding to Havrada and Charvat's [3] probabilistic measure of entropy,

$$\text{H}_{\alpha}(\text{P}) = \frac{1}{1-\alpha} \left(\sum_{i=1}^{n} p_i^{\alpha} - 1 \right); \quad \alpha > 0, \ \alpha \neq 1, \tag{1.5}$$

Kapur [4] suggested the following measure of fuzzy entropy :

$$\mathrm{H}_{\alpha}(\mathrm{A}) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \left[\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha} - 1\right]; \quad \alpha > 0,\ \alpha \neq 1 \tag{1.6}$$

Apparently, there seems to be no relation between trigonometry and fuzzy information theory. Nevertheless one relationship arises as measures of fuzzy entropy used in fuzzy information theory are concave functions and some trigonometric functions are also concave functions. Our interest is to exploit this relationship and establish some inequalities between angles of a convex polygon.

A measure of entropy having these properties and involving trigonometric functions has been given by Kapur and Tripathi [5]. This measure is given by

$$S_2(P) = \sum_{i=1}^{n} \sin \pi p_i \tag{1.7}$$

Corresponding to (1.7), Kapur [4] gave measure of fuzzy entropy as

$$H_2(A) = \sum_{i=1}^{n} \left[\sin \pi\mu_A(x_i) + \sin \pi(1-\mu_A(x_i))\right] \tag{1.8}$$

$$= 2\sum_{i=1}^{n} \sin \pi\mu_A(x_i)$$

Another measure of entropy having same properties and involving logarithmic function is given by

$$S_3(P) = \sum_{i=1}^{n} \log p_i \tag{1.9}$$

Corresponding to (1.9), Parkash and Sharma [6] gave measure of fuzzy entropy as

$$H_3(A) = \sum_{i=1}^{n} \left[\log \mu_A(x_i) + \log\ 1-\mu_A(x_i)\right] \tag{1.10}$$

2. MEASURES OF FUZZY ENTROPY AND THEIR VALIDITY

We, now propose new measures of fuzzy entropy as follows:

$$H_4(A) = \sum_{i=1}^{n} \left[\sin\{\frac{2(n-2)\beta\pi\mu_A(x_i)}{n} + \alpha\} + \sin\{\frac{2(n-2)\beta\pi(1-\mu_A(x_i))}{n} + \alpha\}\right]$$

$$-\sum_{i=1}^{n} \left[\sin\alpha + \sin\frac{2(n-2)\beta\pi}{n} + \alpha\right] \tag{2.1}$$

And

$$H_5(A) = \sum_{i=1}^{n} \left[\tan\{\frac{2\beta(n-2)\pi\mu_A(x_i)}{n} + \alpha\} - \tan\{\frac{2\beta(n-2)\pi(1-\mu_A(x_i))}{n} + \alpha\}\right]$$

$$+\sum_{i=1}^{n} \left[\tan\alpha + \tan\{\frac{2(n-2)\beta\pi}{n} + \alpha\}\right] \tag{2.2}$$

(a) Differentiate (2.1) w.r.t. $\mu_A(x_i)$, we get

$$\frac{\partial^2 H_4(A)}{\partial\mu_A^2(x_i)}$$

$$= -\beta^2 (n-2)^2 \pi^2 \sum_{i=1}^{n} \left[\sin\{\frac{2\beta(n-2)\pi\mu_A(x_i)}{n} + \alpha\} + \sin\{\frac{2\beta(n-2)\pi(1-\mu_A(x_i))}{n} + \alpha\} \right]$$

For $0 \prec \frac{2(n-2)\beta\pi\mu_A(x_i)}{n} + \alpha \leq \pi$ and since $0 \leq \sin\theta \leq 1 \quad when \quad 0 \leq \theta \leq 1$

Thus

$\frac{\partial^2 H_4(A)}{\partial \mu_A^2(x_i)} < 0$ and $H_4(A)$ is a concave function of $\mu_A(x_i)$ and its maximum arises when $\mu_A(x_i) = \frac{1}{2}$

Thus we have the following results :

(*i*) $H_4(A)$ is a concave function of $\mu_A(x_i)$.

(*ii*) $H_4(A)$ doesn't change when $\mu_A(x_i)$ is changed to $1- \mu_A(x_i)$.

(*iii*) $H_4(A)$ is an increasing function of $\mu_A(x_i)$ when $0 \leq \mu_A(x_i) \leq 1/2$.

(*iv*) $H_4(A)$ is a decreasing function of $\mu_A(x_i)$ when $\frac{1}{2} \leq \mu_A(x_i) \leq 1$.

(*v*) $H_4(A)$ =0 when $\mu_A(x_i) = 0$ or 1.

Hence $H_4(A)$ is a valid measure of fuzzy entropy.

(b) Differentiate (2.2) w.r.t. $\mu_A(x_i)$, we get

$$\frac{\partial^2 H_5(A)}{\partial \mu_A^2(x_i)} = -2(n-2)^2 \beta^2 \pi^2 \sum_{i=1}^{n} \left[\sec^2\{\frac{2(n-2)\beta\pi\mu_A(x_i)}{n} + \alpha\} \tan\{\frac{2(n-2)\beta\pi\mu_A(x_i)}{n} + \alpha\} \right]$$

$$-2(n-2)^2 \beta^2 \pi^2 \sum_{i=1}^{n} \left[\sec^2\{\frac{2(n-2)\beta\pi(1-\mu_A(x_i))}{n} + \alpha\} \tan\{\frac{2(n-2)\beta\pi(1-\mu_A(x_i))}{n} + \alpha\} \right]$$

For $\mu_A(x_i) < \frac{1}{n-2}$, we have

$$(n-2)\beta\pi\mu_A(x_i) + \alpha < \beta\pi + \alpha < \frac{\pi}{2} \quad \text{if} \quad \beta \prec \frac{1}{2} - \frac{\alpha}{\pi}$$

So that $\frac{\partial^2 H_5(A)}{\partial \mu_A^2(x_i)} < 0$ and $H_5(A)$ is a concave function of $\mu_A(x_i)$

Thus we have the following results :

(*i*) $H_5(A)$ is a concave function of $\mu_A(x_i)$.

(*ii*) $H_5(A)$ doesn't change when $\mu_A(x_i)$ is changed to $1- \mu_A(x_i)$.

(*iii*) $H_5(A)$ is an increasing function of $\mu_A(x_i)$ when $0 \le \mu_A(x_i) \le 1/2$.

(*iv*) $H_5(A)$ is a decreasing function of $\mu_A(x_i)$ when $\frac{1}{2} \le \mu_A(x_i) \le 1$.

(*v*) $H_5(A) = 0$ when $\mu_A(x_i) = 0$ or 1.

Hence $H_5(A)$ is a valid measure of fuzzy entropy.

3. Basic Trignometric Inqualities

Let $A_1, A_2, A_3, \ldots, A_n$ be the angles measured in radians of a convex polygon of n sides, so that

$$A_1 + A_2 + A_{3,} + \ldots\ldots + A_n = (n-2)\pi$$

Let $\mu_A(x_i) = \dfrac{nA_i}{2(n-2)\pi}$, i = 1,2,3,….n.

Now consider the measure of fuzzy entropy

$$H_4(A) = \left[\sin\{\frac{2(n-2)\beta\pi\mu_A(x_i)}{n}+\alpha\}+\sin\{\frac{2(n-2)\beta\pi(1-\mu_A(x_i))}{n}+\alpha\}\right]$$

$$-\sum_{i=1}^{n}\left[\sin\alpha+\sin\frac{2(n-2)\beta\pi}{n}+\alpha\right] \quad (3.1)$$

Where β *and* α are two parameters satisfying $0 < \alpha < \pi$ and $0 < \beta < 1-\dfrac{\alpha}{\pi}$

Now each $A_i < \pi$ since each A_i is an angle of a convex polygon.

$$\frac{2(n-2)\pi\mu_A(x_i)}{n} < \pi \quad \text{or} \quad \mu_A(x_i) < \frac{n}{2(n-2)}$$

$$\frac{2(n-2)\pi\beta\mu_A(x_i)}{n}+\alpha < \beta\pi+\alpha = \pi(\beta+\frac{\alpha}{\pi}) \le \pi$$

Since $H_4(A)$ is a concave function of $\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots, \mu_A(x_n)$ and has maximum value for the most fuzzy set so that

$$H_4(A) \le H_4(\frac{1}{2}) \quad (3.2)$$

$$H_4\ \mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots \mu_A(x_n) \le H_4\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2}\right)$$

$$\sum_{i=1}^{n}\left[\sin\{\frac{2(n-2)\beta\pi\mu_A(x_i)}{n}+\alpha\}+\sin\{\frac{2(n-2)\beta\pi(1-\mu_A(x_i))}{n}+\alpha\}\right]$$

$$\le \sum_{i=1}^{n}\left[\sin\{\frac{(n-2)\beta\pi}{n}+\alpha\}+\sin\{\frac{(n-2)\beta\pi}{n}+\alpha\}\right]$$

$$\sum_{i=1}^{n}\sin(\beta A_i+\alpha) \le n\sin(\frac{\beta(n-2)\pi}{n}+\alpha) \quad (3.3)$$

Moreover the inequality sign in (3.2) holds only when $\mu_A(x_i) = \frac{1}{2}$ for each i so that equality sign in (3.3) holds when $A_1 = A_2 = A_3 = ... = A_n = \frac{(n-2)\pi}{n}$

Inequality (3.3) is our basic inequality involving trigonometric functions of the angles $A_1, A_2, A_3, ..., A_n$ of a convex polygon of n sides. The equality sign in (3.3) will hold when all the angles are equal.

4. Special Cases

The inequality (3.3) represents a triple infinity of inequalities since it involves three parameters n, α and β. Firstly, we can give any integral value $\geq$ 3 to n. Secondly, we can give any real value to α lying between o and π. Thirdly, corresponding to any value of α, we can give any positive value to k less than 1-$\frac{\alpha}{\pi}$.

We can get special inequalities by giving particular values to α, n, β.

Case I N = 3

For a triangle with angles A_1, A_2, A_3, (3.3) gives

$$\sin(\beta A_1 + \alpha) + \sin(\beta A_2 + \alpha) + \sin(\beta A_3 + \alpha) \leq 3\sin(\beta \frac{\pi}{3} + \alpha) \text{ ; } 0 < \beta \leq 1 - \frac{\alpha}{\pi} \quad (3.4)$$

This gives the inequalities

$$\sin A_1 + \sin A_2 + \sin A_3 \leq \frac{3\sqrt{3}}{2}$$

$$\sin A_1/2 + \sin A_2/2 + \sin A_3/2 \leq \frac{3}{2}$$

$$\cos A_1/2 + \cos A_2/2 + \cos A_3/2 \leq \frac{3\sqrt{3}}{2}$$

Case II N = 4

For a quadrilateral with angles A_1, A_2, A_3, A_4, we have

$$\sin(\beta A_1 + \alpha) + \sin(\beta A_2 + \alpha) + \sin(\beta A_3 + \alpha) + \sin(\beta A_4 + \alpha) \leq 4\sin(\beta \frac{\pi}{2} + \alpha),$$

$$0 < \beta \leq 1 - \frac{\alpha}{\pi}$$

This gives the inequalities

$$\sin A_1 + \sin A_2 + \sin A_3 + \sin A_4 \leq 4$$

$$\sin {A_1}/{2} + \sin {A_2}/{2} + \sin {A_3}/{2} + \sin {A_4}/{2} \leq 2\sqrt{2}$$

$$\sin {A_1}/{3} + \sin {A_2}/{3} + \sin {A_3}/{3} + \sin {A_4}/{3} \leq 2$$

$$\cos {A_1}/{3} + \cos {A_2}/{3} + \cos {A_3}/{3} + \cos {A_4}/{3} \le 2\sqrt{2}$$

Since $H_5(A)$ is a concave function of $\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots, \mu_A(x_n)$ and has maximum value for the most fuzzy set so that

$$H_5(A) \le H_5({1}/{2})$$

that is

$$H_5\ \mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \ldots \mu_A(x_n) \le H_5\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$$

so that

$$\sum_{i=1}^{n}\left[\tan\{\frac{2\beta(n-2)\pi\mu_A(x_i)}{n} + \alpha\}\right] \ge n\tan\{\frac{\beta(n-2)\pi}{n} + \alpha\}$$

Or

$$\sum_{i=1}^{n}\tan(\beta A_i + \alpha) \ge n\tan\{\frac{\beta(n-2)\pi}{n} + \alpha\} \quad (3.5)$$

This is our basic inequality involving tangents of angles of a convex polygon.

Again the inequality (3.5) gives a triple infinity of inequalities since it involves three parameters n, α and β,. The parameter n can take all integral values ≥ 3 . α can take all real values less than π and β can take all real values less than $\frac{1}{2} - \frac{\alpha}{\pi}$.

Particular case

For α =0

$$\sum_{i=1}^{n}\tan \beta A_i \ge n\tan(n-2)\beta\frac{\pi}{n}; \quad \beta < \frac{1}{2}$$

For triangle, $A_1 + A_2 + A_3 = \pi$, n=3, $\beta = \frac{1}{3}$ so that

$$\tan {A_1}/{3} + \tan {A_2}/{3} + \tan {A_3}/{3} \ge 3\tan {\pi}/{9}$$

For quadrilateral, $\tan {A_1}/{4} + \tan {A_2}/{4} + \tan {A_3}/{4} + \tan {A_4}/{4} \ge 4\tan {\pi}/{8}$

5. Conclusion

Minimum Area of a Triangle with Given Perimeter

Although there are many proofs of the result that the area of a triangle with given perimeter is maximum when the triangle is equilateral. Also we have many proofs of the result that the perimeter of a triangle with area is minimum when the triangle is equilateral. Here we give proofs of these results by making use of Fuzzy information theoretic approach

From Hero's formula

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Where a,b,c are the lengths of the sides of the triangle, so that

$$\log \Delta = \frac{1}{2}\ \log s + \log(s-a) + \log(s-b) + \log(s-c)$$

$$= \frac{1}{2}\left[\log s + \log\frac{a(s-a)}{s^2} + \log\frac{b(s-b)}{s^2} + \log\frac{c(s-c)}{s^2} - \log a - \log b - \log c + 3\log s^2\right]$$

$$= \frac{1}{2}\left[7\log s - \log abc + \log\frac{a}{s} + \log\frac{b}{s} + \log\frac{c}{s} + \log\frac{s-a}{s} + \log\frac{s-b}{s} + \log\frac{s-c}{s}\right]$$

$$= \frac{1}{2}\left[\log\frac{s^7}{abc} + \log\mu_A(x_1) + \log\{1-\mu_A(x_1)\} + \log\mu_A(x_2) + \log\{1-\mu_A(x_2)\} + \log\mu_A(x_3) + \log\{1-\mu_A(x_3)\}\right]$$

$$= \frac{1}{2}\log\frac{s^7}{abc} + \frac{1}{2}\sum_{i=1}^{3}\left[\log\mu_A(x_i) + \log\{1-\mu_A(x_i)\}\right]$$

$$= \frac{1}{2}\log\frac{s^7}{abc} - \frac{1}{2}H_3(A),$$

where $H_3(A)$ is fuzzy entropy corresponding to Shannon's[8] probabilistic measure of entropy . and $\mu_A(x_1) = \frac{a}{s}$, $\mu_A(x_2) = \frac{b}{s}$, $\mu_A(x_3) = \frac{c}{s}$, $\sum_{i-1}^{3}\mu_A(x_i) = \frac{a+b+c}{s} = \frac{2s}{s} = 2$

Now if S is given $\log\Delta$ is maximum when $H_3(A)$ is maximum. Also $H_3(A)$ is maximum when $\mu_A(x_1) = \mu_A(x_2) = \mu_A(x_3)$), that is, when a=b=c

Again if Δ is given, $\log\Delta$ is minimum when $H_3(A)$ is maximum that is when the triangle is equilateral.

Thus we have proved both the results stated above.Now for a general quadrilateral, a formula like (4.1) is not available. However, for cyclic quadrilaterals, we have Brahmagupta`s formula

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)} \quad , \quad s = \frac{a+b+c+d}{2}$$

$$\log\Delta = \frac{1}{2}\ \log(s-a) + \log(s-b) + \log(s-c) + \log(s-d)$$

$$= \frac{1}{2}\left[\log\frac{a(s-a)}{s^2} + \log\frac{b(s-b)}{s^2} + \log\frac{c(s-c)}{s^2} + \log\frac{d(s-d)}{s^2} - \log abcd + 4\log s^2\right]$$

$= \frac{1}{2}\left[\log\frac{s^8}{abcd} - H_4(A)\right]$, where $H_4(A)$ is fuzzy entropy corresponding to Shannon's[8] probabilistic measure of fuzzy entropy

That is out of all cyclic quadrilaterals with a given perimeter, the square has maximum area and out of all cyclic quadrilaterals with a given area , the square has a minimum perimeter.

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