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On $(N, p_n, q_n)(C, \alpha, \beta)$ Product Summability of Fourier Series

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Abstract: *In this paper, a theorem on* $(N, p_n, q_n)(C, \alpha, \beta)$ *product summability of Fourier series has been estabilished..*

Keywords: (N, p_n, q_n) *-mean,* (C, α, β) *- mean,* $(N, p_n, q_n)(C, \alpha, \beta)$ *-product mean and Fourier series.*

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1. Introduction

Let Σa_n be a given infinite series with sequence of its nth partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequences of non-negative, non increasing real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty . \tag{1.1}$$

For a positive real sequence $q = \{q_n\}$, we define an increasing sequence $\{r_n\}$ such that

$$r_n = (p * q)_n = \sum_{v=0}^{n} p_{n-v} q_v \to \infty \text{ as } n \to \infty \tag{1.2}$$

denotes the convolution product where

$$Q_n = \sum_{v=0}^{n} q_v, Q_{-i} = q_{-i} = 0, \quad \forall\ i \geq 1 \tag{1.3}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{r_n} \sum_{v=0}^{n} p_{n-v} q_v s_v \tag{1.4}$$

defences the sequence $\{t_n\}$ of the $| N, p_n, q_n |$- mean of the sequence $\{s_n\}$, (Borwein [2]).

If $t_n \to s$ as $n \to \infty$, then the series Σa_n is said to be $| N, p_n, q_n |$-summable to s.

Again let Σa_n be a given infinite series with partial sum $\{s_n\}$ and $t_n^{\alpha,\beta}$ denotes the n[th]cesaro mean of order (α, β) with $\alpha + \beta > -1$ of the sequence $\{s_n\}$ such that,

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{\infty} A_{n-v}^{\alpha-1} s_v \tag{1.5}$$

where $A_n^{\alpha+\beta} = O(n^{\alpha+\beta})$ and $A_0^{\alpha+\beta} = 1$.

If $t_n^{\alpha,\beta} \to s$ as $n \to \infty$. Then the series Σa_n is said to be (C,α,β) summable to s. The product of (N, p_n, q_n)-summability with (C,α,β)-summability defines (N, p_n, q_n) (C,α,β) summability and denoted by $N_{pq}C_n^{\alpha,\beta}$ and

$$\text{If} \quad N_{pq}C_n^{\alpha,\beta} = \frac{1}{r_n}\sum_{k=0}^{n}\frac{p_{v-k}.q_k}{A_k^{\alpha+\beta}}\sum_{v=0}^{k}A_{k-v}^{\alpha-1}A_v^{\beta}s_v \to s \text{ as } n\to\infty \tag{1.6}$$

Then the series Σa_n is said to summable to s by $(N, p_n, q_n)(C,\alpha,\beta)$ -summability method.

In the case when $\beta = 1$ and $q_n = 1\ \forall_n \in N$, then the method $(N, p_n, q_n)(C,\alpha,\beta)$ reduces to (N, p_n) (C,α) and if $p_n = 1 \forall n \in N$ and, $\beta = 1$ then the method (N, p_n, q_n) reduces to $(\bar{N}, q_n)(C,\alpha)$ method. it is known (N, p_n, q_n) and (C,α,β) methods are regular (Hardy [3]).

It is suppose that $(N, p_n, q_n)(c,\alpha,\beta)$ is regular throughout this paper.

Let $f(t)$ be a periodic function with period 2π, integrable in the sence of Lebesgue over $(-\pi,\pi)$ then

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) \tag{1.7}$$

Is the Fourier series associated with f.

We use the following notation throughout this paper.

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$K_n(t) = \frac{1}{2\pi r_n}\sum_{n=0}^{n}\frac{p_{n-k}q_k}{A_k^{a+\beta}}\sum_{v=o}^{k}A_{k-v}^{\alpha-1}A_v^{\beta}\,\frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}}$$

2. Known Result

Dealing with $(N, p_n, q_n)(E,z)$ summability method of a Fourier series,Padhy et al [4] estabi - lished the following theorem.

Theorem 2.1

Let $\{p_n\},\{q_n\}$ and $\{r_n\}$ be sequences satisfying (1.2), (1.2) and

$$\phi(t) = \int_0^t |\phi(u)|\,du = O\left\{\frac{t}{\alpha(1/t)}\right\} \text{ as } t \to +O \tag{2.1}$$

And $a(n) \to \infty$ as $n \to \infty$ (2.2)

where $\alpha(t)$ is positive, non increasing function of t, then the Fourier series $\sum_{n=1}^{\infty} A_n(t)$ is summable $(N, p_n, q_n)(E, Z)$ at the point t.

3. MAIN RESULT

In this paper, we have estabilished a theorem on $(N, p_n, q_n)(C, \alpha, \beta)$ product summability of Fourier series.

Theorem 3.1. Let $\{p_n\}, \{q_n\}$ and $\{r_n\}$ be sequences satisfying (1.1), (1.2) and

$$\phi(t) = \int_0^t |\phi(\mathrm{u})|\, du = \mathrm{O}\left\{\frac{t}{\alpha(1/t)}\right\}, \text{ as } t \to +\mathrm{O} \tag{3.1}$$

And $\alpha(n) \to \infty$ as $n \to \infty$ (3.2)

where $\alpha(t)$ be a positive, non-increasing function of t.

The Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is summable $(N, p_n, q_n)\ (C, \alpha, \beta)$ at the point t.

4. REQUIRED LEMMA

We have required the following lemmas to prove the theorem.

Lemma 4.1

$$|k_n(t)| = \mathrm{O}(n), \mathrm{O} \le t \le \frac{1}{n+1}$$

Proof:

For $0 \le t \le \frac{1}{n+1}$, we have (Boose [1])

$$\sin nt \le n \sin t \text{ and} \sum_{v=0}^{n} A_{k-v}^{\alpha-1} A_v^{\beta} = A_k^{\alpha+\beta}$$

Then

$$|k_n(t)| \le \frac{1}{2\pi r_n}\left|\sum_{k=o}^{n} \frac{p_{n-k} q_k}{A_k^{\alpha+\beta}} \sum_{v=0}^{k} \frac{A_{k-v}^{\alpha-1} A_v^{\beta} (2v+1) \sin\frac{t}{2}}{\sin\frac{t}{2}}\right|$$

$$\le \frac{1}{2\pi r_n}\left|\sum_{k=o}^{n} \frac{p_{n-k} q_k}{A_k^{\alpha+\beta}} (2k+1) \sum_{v=0}^{k} A_{k-v}^{\alpha-1} A_v^{\beta}\right|$$

$$= \frac{1}{2\pi r_n}\left|\sum_{k=o}^{n} \frac{p_{n-k} q_k}{A_k^{\alpha+\beta}} (2k+1) A_k^{\alpha+\beta}\right|$$

$$= \frac{(2n+1)}{2\pi r_n} |\Sigma p_{n-k} q_k|$$

$$= \mathrm{O}(n)$$

Lemma 4.2

$$|k_n(t)| = \mathrm{O}(1/t), \text{ for } 1/n \le t \le \pi$$

Proof:

For $\frac{1}{n} \leq t \leq \pi$, we have by Jordon's lemma,

$\sin(t/2) \geq, \ (t/\pi), \ \ \sin nt \leq 1.$

Then

$$| k_n(t) | \leq \frac{1}{2\pi r_n} \sum_{k=0}^{n} \frac{p_{n-k} q_k}{A_k^{\alpha+\beta}} \left| \sum_{v=0}^{k} A_{k-v}^{\alpha-1} A_v^{\beta} \frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}} \right|$$

$$\leq \frac{1}{2\pi r_n} \sum_{k=0}^{n} \frac{p_{n-k} q_k}{A_k^{\alpha+\beta}} \left| \sum_{v=0}^{k} A_{k-v}^{\alpha-1} A_v^{\beta} \left(\frac{\pi}{t} \right) \right|$$

$$= \frac{1}{2\pi r_n} \sum_{k=0}^{\infty} \frac{p_{n-k} q_k}{A_k^{\alpha+\beta}} A_k^{\alpha+\beta}$$

$$= \mathrm{O}(1/t)$$

5. Proof of the Theorem

If $s_n(f;x)$ is the n-*th* partial sum of the Fourier series $\sum_{n=0}^{\infty} A_n(t)$ of $f(t)$ then by using Riemann–Lebesgue theorem, we have (Titchmarch [5]).

$$s_n(f;x) - f(x) = \frac{1}{2\pi} \int_0^{\pi} \phi(t) \frac{\sin(n+\frac{1}{2})t}{\sin(\frac{t}{2})} dt$$

If $N_{pq} C_n^{\alpha,\beta}$ denote the $(N, p_n, q_n)(C, \alpha, \beta)$ transform of $s_n(f;x)$, we have

$$N_{pq} C_n^{\alpha,\beta} - f(x) = \frac{1}{2\pi r_n} \sum_{k=0}^{n} \frac{p_{n-k} q_k}{A_k^{\alpha+\beta}} \int_0^{\pi} \frac{\phi(t)}{\sin(t/2)} \left\{ \sum_{v=0}^{k} A_{k-v}^{\alpha-1} A_v^{\beta} \sin(v+\tfrac{1}{2})t \right\} dt$$

$$= \int_0^{\pi} \phi(t) k_n(t) dt$$

In order to prove the theorem, it is sufficient to show that

$$\int_0^{\pi} \phi(t) k_n(t) dt = \mathrm{O}(1) \text{ as } n \to \infty$$

For $0 < \delta < \pi$, we have

$$N_{pq} C_n^{\alpha,\beta} - f(x) = \int_0^{\pi} \phi(t) k_n(t) dt$$

$$= \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} \right) \phi(t) k_n(t) dt$$

$$= I_1 + I_2 + I_3 \qquad \text{(say)}$$

Now

$$| I_1 | = \left| \int_0^{1/n} \phi(t) k_n(t) dt \right|$$

$$\leq \int_0^{1/n} |\phi(t)||k_n(t)|\,dt$$

$$\leq \mathrm{O}(n)\left\{\mathrm{O}\left(\frac{1}{n\alpha(n)}\right)\right\}$$

$$= \mathrm{O}\left(\frac{1}{\alpha(n)}\right) \quad \text{as } n \to \infty$$

$$= \mathrm{O}(1) \quad \text{as } n \to \infty$$

Next

$$|I_2| = \left|\int_{1/n}^{\delta} \phi(t)k_n(t)dt\right|$$

$$\leq \int_{1/n}^{\delta} |\phi(t)||k_n(t)|\,dt$$

$$\leq \mathrm{O}\left\{\int_{1/n}^{\delta} \frac{|\phi(t)|}{t}dt\right\}$$

$$\text{-O}\left\{\left[\frac{\Phi(t)}{t}\right]_{1/n}^{\delta} + \int_{1/n}^{\delta} \frac{\Phi(t)}{t^2}dt\right\}$$

$$= \mathrm{O}\left\{\mathrm{O}\left[\frac{1}{\alpha(1/t)}\right]_{1/n}^{\delta} + \int_{1/\delta}^{n} \mathrm{O}\left(\frac{1}{u\alpha(u)}\right)\right\}du$$

Where $u = \frac{1}{t}$ and $0 < \delta < 1$

$$= \mathrm{O}\left(\frac{1}{a(n)}\right) + \mathrm{O}\left(\frac{1}{n\alpha(n)}\right)\int_{1/\delta}^{n} du$$

Using second mean value theorem for the integral in the 2nd term as $\alpha(n)$ is monotonic

$$= \mathrm{O}(1) + \mathrm{O}(1), \quad \text{as } n \to \infty$$

$$= \mathrm{O}(1) \quad \text{as } n \to \infty$$

Finally

$$|I_3| \leq \int_{\delta}^{\pi} |\phi(t)||k_n(t)|\,dt$$

$$= \mathrm{O}(1) \quad \text{as} \quad n \to \infty$$

by using Riemann-Lebesgue theorem and the regularity condition of the method of summability.

Thus $N_{pq}C_n^{\alpha,\beta} - f(x) = \mathrm{O}(1)$ as $n \to \infty$

This completes the proof of the theorem.

6. Conclusion

In this paper a more general result for summability of Fourier series is established which will be enrich the Literature of Fourier series.

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