International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 1, January - 2014, PP 1-15
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Some Elementary Problems From the Note Books of Srinivasa Ramanujan-Part(II)

N.Ch. Pattabhi Ramacharyulu

Professor (Rtd) NIT Warangal
Warangal-506004
pattabhi1933@yahoo.com

§§ Problem 13: **SRMs (189) p and NBSR Vol. II p 193**

Let $\prod(a,x) = (1 + a)(1 + ax)(1 + ax^2)(1 + ax^3)(1 + ax^4)$ . . . . and so on --------------------- (13.1)

Then (i) $\dfrac{\prod(a,x)}{\prod(ax^n,x)} = (a + x)^n$ when x = 1 (*.1)

(ii) $\dfrac{\prod(-x,x)}{(1-x)^n\prod(-x^{n+1},x)} = n!$ where x = 1 (*.2)

(iii) $\prod(a,x) = \dfrac{\prod(a,\sqrt{x})}{\prod(a\sqrt{x},x)}$ (*.3)

Solution:

Note. *S.R.denotes* $\prod(a,x)$ *as the product of infinite number of terms of the type ($1 + ax^k$)*

i.e., $\prod(a,x) = \prod_{k=m}^{\infty}(1+ax^k)$

(i) To establish the result (*.1)

$$\prod(ax^n,x) = (1 + ax^n)(1 + ax^n.x)(1 + ax^n.x^2)(1 + ax^n.x^3) \text{ - - - - - - - - - - - -}$$
$$= (1 + ax^n)(1 + ax^{n+1})(1 + ax^{n+2})(1 + ax^{n+3}) \text{ - - - - - - - - - - - -}$$
$$= \prod_{k=0}^{\infty}(1+ax^k) \qquad (13.2)$$

By definition (1)

$$\prod(a,x) = (1 + a)(1 + ax)(1 + ax^2)(1 + ax^3)\ldots(1 + ax^{n-1})(1 + ax^n)(1 + ax^{n+1})\ldots\ldots$$
$$= (1 + a)(1 + ax)(1 + ax^2)(1 + ax^3)\ldots\ldots(1 + ax^{n-1})\prod_{k=m}^{\infty}(1+ax^k) \qquad (13.3)$$
$$= (1 + a)(1 + ax)(1 + ax^2)(1 + ax^3)\ldots\ldots(1 + ax^{n-1})\prod(ax^n,x)$$

$$\therefore \frac{\prod(a,x)}{\prod(ax^n,x)} = (1 + a)(1 + ax)(1 + ax^2)(1 + ax^3)\ldots\ldots(1 + ax^{n-1}) \qquad (13.4)$$

When x = 1, R.H.S. of (4) = (1 + a) (1 + a) (1 + a) (1 + a) . . . . . . (1 + a) (n times)
$= (1 + a)^n$

This establishes the result (*.1)

(ii) To establish the result (*.2)

$$\prod(-x,x) = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)\ldots\ldots = \prod_{k=1}^{\infty}(1-x^k)$$

and $\prod(-x^{n+1}, x) = (1 - x^{n+1})(1 - x^{n+1}.x)(1 - x^{n+1}.x^2)(1 - x^{n+1}.x^3)\ldots\ldots = \prod_{k=n+1}^{\infty}(1 - x^k)$

$$\therefore \quad \frac{\prod(-x, x)}{(1-x)^n \prod(-x^{n+1}, x)} = \frac{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots\ldots(1-x^n)(1-x^{n+1})\ldots\ldots}{(1-x)^n(1-x^{n+1})(1-x^{n+2})\ldots\ldots}$$

$$= \frac{(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots\ldots(1-x^n)}{(1-x)^n}$$

$$= \frac{(1-x)}{(1-x)}\cdot\frac{(1-x^2)}{(1-x)}\cdot\frac{(1-x^3)}{(1-x)}\ldots\ldots\frac{(1-x^n)}{(1-x)}$$

$$= 1.\,(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)\ldots \text{ factors} \qquad (13.5)$$

The last factor on the R.H.S. of the above expression = $1+x+x^2+x^3+\ldots+x^{n-1}$ (n terms)

When x = 1, R.H.S. of (13.5) = 1.2.3.4. . . . . .n = n!

This establishes the result (*.2)

(iii) To establish the result (*.3) By definition (1), we have

$$\prod(a\sqrt{x}, x) = (1 + a\sqrt{x})(1 + a\sqrt{x}.x)(1 + a\sqrt{x}.x^2)(1 + a\sqrt{x}.x^3)\ldots\ldots$$

$$\prod(a, \sqrt{x}) = (1 + a)(1 + ax^{1/2})(1 + ax)(1 + ax^{3/2})(1 + ax^2)(1 + ax^{5/2})(1 + ax^3)\ldots\ldots$$

$$= \{(1 + a)(1 + ax)(1 + ax^2)\ldots\ldots\} \times \{(1 + a\sqrt{x})(1 + ax\sqrt{x})(1 + ax^2\sqrt{x})(1 + ax^3\sqrt{x})\ldots\ldots\}$$

$$= \prod(a, x).\prod(a\sqrt{x}, x)$$

$$\therefore \quad \frac{\prod(a, \sqrt{x})}{\prod(a\sqrt{x}, x)} = (1 + a)(1 + ax)(1 + ax^2)(1 + ax^3)\ldots\ldots = \prod(a, x)$$

This result (*.3) is established.

-- -----@@@@--

Problem 14: **SRMs (1) P 105,** NBSR **Vol. I p 209 and Vol. II p 143**

$$\sqrt{ax + (n+a)^2 + x\sqrt{a(x+n) + (n+a)^2 + (x+n)\sqrt{\ldots\ldots\ldots\ldots}}} \qquad (*)$$

Note. (i) This fascinating representation of an integer as a nested square root expansion finds a place in the collected works of S.R. (p.323)

(ii) This was a problem proposed by S.R. in the Journal of Indian Mathematical Society (problem No. 289, Vol.III (   ) p.90). The problem created an astonishment amongst the problem solvers on the land. “An integer can be expressed as a nested square root assembly – on the face of it an irrational ! How is this possible !!”

The problem was referred back by the Editorial Board of J.I.M.S. to S.R. for clarification and to provide the solution. S.R. responded by giving the solution to the problem and also provided examples such as following.

Solution: Let { L.H.S. of *}^{2} = f (x) (say) ----------------------------- (1)

i.e., $f(x) = (x + n + a)^2$ (Expanding the R.H.S. of (1) and rearranging the terms)

$= ax + (n + a)^2 + x\{(x + n) + (n + a)\}$

$$= ax + (n+a)^2 + x\sqrt{f(x+n)} \qquad \text{-------------------------} \qquad (2)$$

This result (2) may be construed as a iterative formula for expressing f (x + r)

$$\therefore \{\text{L.H.S. of } *\} = \sqrt{f(x)} = \sqrt{\mathbf{ax} + (\mathbf{n}+\mathbf{a})^2 + \mathbf{x}\sqrt{f(x+n)}} \qquad \text{using (2)}$$

$$= \sqrt{\mathbf{ax} + (\mathbf{n}+\mathbf{a})^2 + \mathbf{x}\left\{\sqrt{a(x+n)+(n+a)^2+(x+n)\sqrt{f(x+2n)}}\right\}}$$

$$= \sqrt{ax+(n+a)^2+x\sqrt{\left\{a(x+n)+(n+a)^2+(x+n)\sqrt{a(x+2n)+(n+a)^2+(x+2n)\sqrt{f(x+3n)}}\right\}}}$$

$$= [\mathbf{ax} + (\mathbf{n}+\mathbf{a})^2 + \mathbf{x}\{a(x+n)+(n+a)^2+(x+n)\langle a(x+2n)$$

$$+(n+a)^2+(x+2n)(a(x+3n)+(\mathbf{n}+\mathbf{a})^2$$

$$+(\mathbf{x}+3\mathbf{n})\sqrt{f(x+4n)}\,)^{1/2}\rangle^{1/2}\}^{1/2}]^{1/2} \quad \ldots\ldots\ldots \text{ and so on. R.H.S. of } *.$$

This establishes the result *

Deductions :

(i) When x = 1, n = 1, a = 1, we get

$$3 = \sqrt{(1+2^2)+1.\sqrt{(2+2^2)+2\sqrt{(3+2^2)+3\sqrt{(4+2^2)+4\sqrt{(5+2^2)+5\sqrt{(6+2^2)+}}}}}}$$

and so on

(ii) When x = 2, n = 1, a = 1, we get

$$4 = \sqrt{(2+2^2)+2.\sqrt{(3+2^2)+3\sqrt{(4+2^2)+4\sqrt{(5+2^2)+5\sqrt{(6+2^2)+}}}}} \cdots\cdot$$

and so on

(iii) When a = n = x we get

$$3x = \sqrt{x^2+(2x)^2+x.\sqrt{2x^2+(2x)^2+2x\sqrt{3x^2+(2x)^2+3x\sqrt{4x^2+(2x)^2+4x\sqrt{5x^2+(2x)^2}}}}}$$

. . . . . . and so on.

$$= \sqrt{5x^2 + x.\sqrt{6x^2 + 2x\sqrt{7x^2 + 3x\sqrt{8x^2 + 4x\sqrt{9x^2 + 5x\sqrt{\ldots}}}}}}$$ and so on and many more such results.

(iv) When x = 3, n = 1, a = 1, we get

$$5 = \sqrt{(3+2^2)+3.\sqrt{(4+2^2)+4\sqrt{(5+2^2)+5\sqrt{(6+2^2)+6(7+2^2)+\ldots\ldots\ldots}}}}$$ and so on.

-- -----@@@@---

Problem 15 **NBSR Vol. p.**

$$\S\ 3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \&\ conti \ldots\ldots}}}}} \qquad (*)$$

Solution : $3 = \sqrt{9}$

$= \sqrt{1 + 2\sqrt{16}}$	$9 = 1 + 8 = 1 + 2x4 = 1 + 2\sqrt{16}$
$= \sqrt{1 + 2\sqrt{1 + 3\sqrt{25}}}$	$16 = 1 + 15 = 1 + 3x5 = 1 + 3\sqrt{25}$
$= \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{36}}}}$	$25 = 1 + 24 = 1 + 4x6 = 1 + 4\sqrt{36}$
$= \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{49}}}}}$	$36 = 1 + 35 = 1 + 5x7 = 1 + 5\sqrt{49}$
$= \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + 6\sqrt{64}}}}}}$	$49 = 1 + 48 = 1 + 6x8 = 1 + 6\sqrt{64}$

. . . . and so on.

= Result of *

NOTE : (1) Observe the occurrences of square numbers at the right end of each step.

(2) This example appeared as a problem in the William Lower Putnam Competition in the year 1966 and also in many talent tests conducted at National and International levels during the last four decades.

(3) The representation * of 3 as nested square roots is not unique. The representation changes with the partitioning of 9 (= 3^2) as illustrated here under. As a matter of fact, this non – uniqueness of the nested root representation is true for any number.

ILLUSTRATION: I.

$3 = \sqrt{9}$ $\qquad (\because 9 = 6 + 3 = 6 + \sqrt{9}\ )$

$= \sqrt{6 + \sqrt{9}}$

$= \sqrt{6 + \sqrt{6 + \sqrt{9}}}$

$= \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{9}}}}}$ and so on.

ILLUSTRATION : II.

$3 = \sqrt{9}$

$= \sqrt{3 + 2\sqrt{9}} \;=\; \sqrt{3 + 2\sqrt{3 + 2\sqrt{9}}}$ $\qquad 9 = 3 + 6 = 3 + 2\sqrt{9}$

$= \sqrt{3 + 2\sqrt{3 + 2\sqrt{3 + 2\sqrt{3 + 2\sqrt{3}}}}}$ . . . . and so on.

ILLUSTRATION : III.

$3 = \sqrt{9}$

$= \sqrt{5 + 2\sqrt{4}}$ $\qquad 9 = 5 + 4 = 5 + 2\sqrt{4}$

$= \sqrt{5 + 2\sqrt{2 + \sqrt{4}}} \;=\; \sqrt{5 + 2\sqrt{2 + \sqrt{2 + \sqrt{4}}}}$ $\qquad 4 = 2 + \sqrt{4}$

$= \sqrt{5 + 2\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{4}}}}}}$ . . . . and so on.

-- -----@@@@---

Problem 16:

§ $4 = \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{9 + \cdots \ldots \ldots \ldots}}}}$ $\qquad$ (*)

Solution: $4 = \sqrt{16} = = \sqrt{6 + 2\sqrt{25}}$ $\qquad (\because 16 = 6 + 10 = 6 + 2\sqrt{25}$ and $25 = 7 + 18 = 7 + 3\sqrt{36}\,)$

$$= \sqrt{6 + 2\sqrt{7 + 3\sqrt{\mathbf{36}}}}$$

$36 = 8 + 28 = 8 + 4\sqrt{\mathbf{49}}$

$$= \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{\mathbf{49}}}}}$$

$49 = 9 + 40 = 9 + 5\sqrt{\mathbf{64}}$

$$= \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{9 + 5\sqrt{\mathbf{64}}}}}}$$

$64 = 10 + 54 = 10 + 6\sqrt{\mathbf{81}}$

$$= \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{9 + 5\sqrt{10 + 6\sqrt{\mathbf{81}}}}}}}$$

$81 = 11 + 70 = 11 + 7\sqrt{\mathbf{100}}$

$$= \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{9 + 5\sqrt{10 + 6\sqrt{11 + 7\sqrt{\mathbf{100}}}}}}}} \quad \ldots\ldots\ \text{and so on.}$$

-- -----@@@@--

Problem 17 : **PROBLEM No. 289 suggested by S.R. to JIMS Vol. III p 30**

§ $\mathbf{n(n+2) = n\sqrt{1 + (n+1)\sqrt{1 + (n+2)\sqrt{1 + (n+3)\sqrt{1 + (n+4)\sqrt{1 + \ldots\ldots}}}}}}$ and so on. *

Solution:

Let f(n) = n(n+2)

$$= n\sqrt{(n+2)^2}$$

$$= n\sqrt{1 + f(n+1)}$$

$(n+2)^2 = = 1 + (n+1)(n+2) = 1 + f(n+1)$

This relation can be taken as recurrence formula for f(n)

$$\therefore f(n) = n\sqrt{1 + (n+1)f(n+2)}$$

$$= n\sqrt{1 + (n+1)\sqrt{1 + (n+2)\sqrt{1 + (n+3)\sqrt{1 + f(n+4)}}}} \quad \text{and so on.}$$

Hence

n(n+2) = $\mathbf{n}\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+(n+4)\sqrt{1+\ldots\ldots\ldots}}}}}$ and so on.

Special Cases :

(i) When n = 1 , L.H.S. * = 3, then

$$3 = 1\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\ldots\ldots\ldots\ldots\ldots\ldots \textit{and so on.}}}}}$$

(ii)When n = 2, L.H.S. * = 8, then

$$8 = 2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\sqrt{1+6\ldots\ldots\ldots\ldots\ldots\ldots \textit{and so on.}}}}}}$$

$$\therefore 4 = \sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\ldots\ldots\ldots\ldots\ldots\ldots \textit{and so on.}}}}}$$

--@@@@--

Problem 18 : **NBSR**

§ **n(n + 3) = n** $\sqrt{(n+5)+(n+1)\sqrt{(n+6)+(n+2)\sqrt{(n+7)+\cdots\ldots\ldots\ldots\ldots}}}$ and so on. *

Solution:

Let f(n) = n(n+3)

$= n\sqrt{(n+3)^2}$ { ∵ $(n+3)^2$ = (n+5) + (n+1)(n+4) = n+5 + f(n+1)}

$= n\sqrt{(N+5)+f(n+1)}$

a recurrence formula for f(n) = n(n+3)

∴ n(n+3) = $n\sqrt{(n+5)+f(n+1)}$

$= n\sqrt{(n+5)+(n+1)\sqrt{(n+6)+f(n+2)}}$

$= n\sqrt{(n+5)+(n+1)\sqrt{(n+6)+(n+2)\sqrt{(n+7)+f(n+3)}}}$ and so on.

$= n\sqrt{(n+5)+(n+1)\sqrt{(n+6)+(n+2)\sqrt{(n+7)+(n+3)\sqrt{(n+8)+\ldots}}}}$ and so on.

This establish the result *

Corollary: Cancelling the common factor n on both the sides of * , we have

$$(n+3) = \sqrt{(n+5)+(n+1)\sqrt{(n+6)+(n+2)\sqrt{(n+7)+(n+3)\sqrt{(n+8)+\ldots}}}}$$

Deductions: **(i)** When n = 1, we get

$$4 = \sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9}+\ \ldots\ldots\ldots\ldots}}}$$ and so on.

(ii) When n =2, we get

$$5 = \sqrt{7+3\sqrt{8+4\sqrt{9+5\sqrt{10}+\ \ldots\ldots\ldots\ldots}}}$$ and so on.

(iii) When n = 3, we get

$$6 = \sqrt{8+4\sqrt{9+5\sqrt{10+}\ \ldots\ldots\ldots\ldots}}$$ and so on.

-- -----@@@@--

Problem 19 : **NBSR Vol.II P.143**

§ $2\cos\theta = \sqrt{2+2\cos 2\theta} = \sqrt{2+\sqrt{2+2\cos 4\theta}} = \sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}}$ *

Proof :

$2\cos\theta = \sqrt{4\cos^2\theta}$ [$\because 4\cos^2\theta = 2.2\cos^2\theta = 2(1+\cos 2\theta) = 2 + 2\cos 2\theta$]

$= \sqrt{2+2\cos 2\theta}$

$= \sqrt{2+\sqrt{4\cos^2 2\theta}}$ [$\because 4\cos^2 2\theta = 2 + 2\cos 4\theta$]

$= \sqrt{2+\sqrt{2+2\cos 4\theta}}$

$= \sqrt{2+\sqrt{2+\sqrt{4\cos^2 4\theta}}}$ [$\because 4\cos^2 4\theta = 2 + 2\cos 8\theta$]

$= \sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta)}}}$ This establishes the results shown by *

NOTE:

(i) Repeating the above process n times, we get

$$2\cos\theta = \sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots\ldots\ldots+\sqrt{2+2\cos(2^n\theta)}}}}$$

(ii) Repeating the process indefinitely, we get

$$2\cos\theta = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots\ldots\ldots}}}}}}$$ and so on

(iii) When $\theta = 0$, we get

$$2 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots\ldots\ldots}}}}}}$$ and so on

--- -----@@@@--

Problem 20 **NBSR Vol.II P.143**

§ $2\cos\theta = \sqrt[3]{2\cos 3\theta + 3\sqrt[3]{2\cos 3\theta + 3\sqrt[3]{2\cos 3\theta + 3\sqrt[3]{2\cos 3\theta + \ldots\ldots\ldots\ldots\ldots}}}}$

(*.1)

$= \sqrt[3]{6\cos\theta + \sqrt[3]{6\cos 3\theta + \sqrt[3]{6\cos 9\theta + \sqrt[3]{6\cos 27\theta + \ldots\ldots\ldots\ldots\ldots}}}}$ and so on . (*.2)

Proof:

To establish (*.1) :

$2\cos\theta = \sqrt[3]{8\cos^3\theta} = \sqrt[3]{2.4\cos^3\theta}$ ($\because 4\cos^3\theta = \cos 3\theta + 3\cos\theta$)

$= \sqrt[3]{2(\cos 3\theta + 3\cos\theta)}$ . . . . (1)

$= \sqrt[3]{2\cos 3\theta + 3(2\cos\theta)}$

$= \sqrt[3]{2\cos 3\theta + 3\sqrt[3]{8\cos^3\theta}}$

$= \sqrt[3]{2\cos 3\theta + 3\sqrt[3]{2\cos 3\theta + 3\sqrt[3]{8\cos^3\theta}}}$ ($\because 8\cos^3\theta = 2[\cos 3\theta + 3\cos\theta]$)

$= \sqrt[3]{2\cos 3\theta + 3\sqrt[3]{2\cos 3\theta + 3\sqrt[3]{2\cos 3\theta + 3\sqrt[3]{8\cos^3 3\theta}}}}$ ($\because 8\cos^3\theta = 6\cos\theta + \sqrt[3]{8\cos^3 3\theta}$)

$= \sqrt[3]{2\cos 3\theta + 3\sqrt[3]{2\cos 3\theta + 3\sqrt[3]{2\cos 3\theta + 3\sqrt[3]{2\cos 3\theta + \ldots\ldots\ldots\ldots\ldots}}}}$ and so on.

Note :

When $\theta = 0$, we get $2 = \sqrt[3]{2 + 3\sqrt[3]{2 + 3\sqrt[3]{2 + 3\sqrt[3]{2} + 3\sqrt[3]{2} + \ldots\ldots\ldots\ldots}}}$

--- -----@@@@--

§ **Problem 21 :** **Not from NBSR**

2sinθ $= \sqrt{4\sin^2\theta}$ [$\because 4\sin^2\theta = 2.2\cos^2\theta = 2(1-\cos 2\theta) = 2 - 2\cos 2\theta$]

$= \sqrt{2 - 2\cos 2\theta}$

$= \sqrt{2 - \sqrt{4\cos^2 2\theta}}$

$= \sqrt{2 - \sqrt{2(1+\cos 4\theta)}}$

$$= \sqrt{2 - \sqrt{2 + 2\cos 4\theta}}$$

$$= \sqrt{2 - \sqrt{2 + \sqrt{4\cos^2 4\theta}}}$$

$$= \sqrt{2 - \sqrt{2 + \sqrt{2(1 + \cos 8\theta)}}}$$

$$= \sqrt{2 - \sqrt{2 + \sqrt{2 + 2\cos 8\theta)}}}$$

$$= \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos 16\theta)}}}} \; .$$

$$= \sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots\ldots\ldots\ldots + \sqrt{2 + 2\cos(2^n\theta)}}}}$$ = (n – nested square roots)

-----@@@@-----

Problem 22: **NBSR Vol. II p.305**

§ $$\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 - \sqrt{2}}}}}}}} \ldots\ldots\ldots\ldots = 2sin\frac{\pi}{18} = 2\sin 10^0$$

Proof:

$$2sin\frac{\pi}{18} = \sqrt{4sin^2\frac{\pi}{18}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$= \sqrt{2.2sin^2\frac{\pi}{18}}$$

$$= \sqrt{2\left(1 - cos\frac{\pi}{9}\right)}$$ ($\because$ $2\sin^2\theta = 1 - \cos 2\theta$)

$$= \sqrt{2 - \sqrt{4cos^2\frac{\pi}{9}}}$$

$$= \sqrt{2 - \sqrt{2\left(1 + cos\frac{2\pi}{9}\right)}}$$ ($\because$ $2\cos^2\theta = 1 + \cos 2\theta$)

$$= \sqrt{2 - \sqrt{2 + \sqrt{4cos^2\frac{2\pi}{9}}}}$$

$$= \sqrt{2 - \sqrt{2 + \sqrt{2\left(1 + cos\frac{4\pi}{9}\right)}}}$$ ($\because \cos\frac{4\pi}{9} = \sin\left(\frac{\pi}{2} - \frac{4\pi}{9}\right) = \sin\frac{\pi}{18}$)

$$= \sqrt{2 - \sqrt{2 + \sqrt{2 + 2\sin\frac{\pi}{18}}}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \qquad (2)$$

The last term sin in (2) $\frac{\pi}{18}$ is the L.H.S. of (1)

By repeating the steps from (1) to (2) , we obtain

$$2\sin\frac{\pi}{18} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2\ldots\ldots\ldots\ldots}}}}}}}}$$

NOTE: The distribution of the sequence of the signs: **- + + - + + -** is to be noted.

-- -----@@@@---

Problem 23: **NBSR Vol. II p 305**

§ $2\sqrt{3}\cos\frac{\pi}{18}-1=2\sqrt{3}\cos 10^0-1=\sqrt{8-\sqrt{8-\sqrt{8+\sqrt{8-\sqrt{8-\sqrt{8+\sqrt{8-\ldots\ldots}}}}}}}$ and so on- - *

Proof:

$$2\sqrt{3}\cos 10^0-1=\sqrt{\left(2\sqrt{3}\cos 10^0-1\right)^2}$$

$$=\sqrt{12\cos^2 10^0-4\sqrt{3}\cos 10^0+1}$$

$$=\sqrt{6\left(1+\cos 20^0\right)-4\sqrt{3}\cos 10^0+1} \qquad [\because 2\cos^2\theta = 1+\cos 2\theta]$$

$$=\sqrt{7+4\sqrt{3}\cos 30^0\cos 20^0-4\sqrt{3}\cos 10^0}$$

$$[\because 6=2.3=2\sqrt{3}\sqrt{3}=2\sqrt{3}(2\cos 30^0)]$$

$$=\sqrt{7+2\sqrt{3}\left(2\cos 30^0\cos 20^0\right)-4\sqrt{3}\cos 10^0}$$

$$=\sqrt{7+2\sqrt{3}\left(\cos 50^0+\cos 10^0\right)-4\sqrt{3}\cos 10^0}$$

$$=\sqrt{7+2\sqrt{3}\cos 50^0-2\sqrt{3}\cos 10^0} \qquad [\because 2\cos A\cos B=\cos(A+B)+\cos(A-B)]$$

$$=\sqrt{7+2\sqrt{3}(\cos 50^0-\cos 10^0)}$$

$$=\sqrt{7-4\sqrt{3}\sin 30^0\sin 20^0}$$

$$\left[\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$=\sqrt{7-2\sqrt{3}\sin 20^0} \qquad \left[\because \sin 30^0=\frac{1}{2}\right]$$

$$=\sqrt{8-(1+2\sqrt{3}\sin 20^0)}$$

$$= \sqrt{8-\sqrt{\left(1+2\sqrt{3}\sin 20^0\right)^2}}$$

$$= \sqrt{8-\sqrt{1+4\sqrt{3}\sin 20^0+12\sin^2 20^0}}$$

$$= \sqrt{8-\sqrt{1+4\sqrt{3}\sin 20^0+6\left(1-\cos 40^0\right)}}$$

$$= \sqrt{8-\sqrt{7+4\sqrt{3}\sin 20^0-6\cos 40^0}}$$

$$= \sqrt{8-\sqrt{7+4\sqrt{3}\sin 20^0-4\sqrt{3}\cos 30^0\cos 40^0}} \qquad [\because 6 = 4\sqrt{3}\cos 30^0]$$

$$= \sqrt{8-\sqrt{7+4\sqrt{3}\cos 70^0-2\sqrt{3}\left(\cos 70^0+\cos 10^0\right)}}$$

[$\because \sin 20^0 = \cos 70^0$ and $2\cos 30^0\cos 40^0 = \cos 70^0 + \cos 10^0$]

$$= \sqrt{8-\sqrt{7+2\sqrt{3}\left(\cos 70^0-\cos 10^0\right)}}$$

$$= \sqrt{8-\sqrt{7+2\sqrt{3}\left(-2\sin 40^0\sin 30^0\right)}}$$

$$= \sqrt{8-\sqrt{7-2\sqrt{3}\sin 40^0}}$$

$$= \sqrt{8-\sqrt{8-\left(1+2\sqrt{3}\sin 40^0\right)}}$$

$$= \sqrt{8-\sqrt{8-\sqrt{\left(1+2\sqrt{3}\sin 40^0\right)^2}}}$$

$$= \sqrt{8-\sqrt{8-\sqrt{1+4\sqrt{3}\sin 40^0+12\sin^2 40^0}}}$$

$$= \sqrt{8-\sqrt{8-\sqrt{1+4\sqrt{3}\sin 40^0+6\left(1-\cos 80^0\right)}}}$$

$$= \sqrt{8-\sqrt{8-\sqrt{7+4\sqrt{3}\cos 50^0-2\sqrt{3}\left(\cos 110^0+\cos 50^0\right)}}}$$

[$\because \sin 40^0 = \cos 50^0$ and $6\cos 80^0 = 4\sqrt{3}\cos 30^0\cos 80^0 = 2\sqrt{3}(\cos 110^0 + \cos 50^0)$]

$$= \sqrt{8-\sqrt{8-\sqrt{7+2\sqrt{3}\left(\cos 50^0-\cos 110^0\right)}}}$$

$$= \sqrt{8-\sqrt{8-\sqrt{7+2\sqrt{3}.2\sin 80^0 \sin 30^0}}}$$

$$= \sqrt{8-\sqrt{8-\sqrt{7+2\sqrt{3}\cos 10^0}}} \qquad [\because \sin 80^0 = \cos 10^0 \text{ and } 30^0 = \tfrac{1}{2}]$$

$$= \sqrt{8-\sqrt{8-\sqrt{8+\left(2\sqrt{3}\cos 10^0 - 1\right)}}} \quad . . . . (2)$$

The term $(2\sqrt{3}\ \cos 10^0 - 1)$ is on the L.H.S. of **(1)**.

By continued iteration on $(2\sqrt{3}\ \cos 10^0 - 1)$,

we obtain the result.

$$2\sqrt{3}\cos\frac{\pi}{18} - 1 = 2\sqrt{3}\cos 10^0 - 1 = \sqrt{8-\sqrt{8-\sqrt{8+\sqrt{8-\sqrt{8-\sqrt{8+\sqrt{8-\ldots\ldots}}}}}}}$$ and so on.

Remark: The pattern of the signs in the series - - + - - + ... is worth noting.

-----@@@@-----

§ § PROBLEM 24 : **NOT FROM NBSR**

$$3 = \sqrt[3]{11+8\sqrt[3]{5+\sqrt[3]{11+8\sqrt[3]{5+\sqrt[3]{11+8\sqrt[3]{5+\ldots\ldots\ldots}}}}}} \qquad *$$

Solution:

$$3 = \sqrt[3]{27}$$

$$= \sqrt[3]{11+8\sqrt[3]{8}} \qquad 27 = 11 + 8.2 = 11 + 8.\sqrt[3]{8}$$

$$= \sqrt[3]{11+8\sqrt[3]{5+\sqrt[3]{27}}} \qquad 8 = 5 + 3 = 5 + \sqrt[3]{27}$$

Repeating the expression for $\sqrt[3]{27}$ again and again, we get

$$3 = \sqrt[3]{11+8\sqrt[3]{5+\sqrt[3]{11+8\sqrt[3]{5+\sqrt[3]{11+8\sqrt[3]{5+\ldots\ldots\ldots}}}}}}$$ This establishes the result *.

-----@@@@-----

Problem 25: **NBSR Vol. II p 143**

§ $x = a_1 + \sqrt{x^2 + a_1(a_1 - 2a_2) - 2a_1\sqrt{x^2 + a_2(a_2 - 2a_3) - 2a_2\sqrt{x^2 + a_3(a_3 - 2a_4) - \ldots\ldots}}}$ and so on.

Proof:

$$x = a_1 + (x - a_1)$$

$$= a_1 + \sqrt{(x - a_1)^2} \quad . . . . (1)$$

$$= a_1 + \sqrt{x^2 + a_1^2 - 2xa_1}$$

$$= a_1 + \sqrt{x^2 + a_1(a_1 - 2a_2) - 2a_1 x + 2a_1 a_2}$$

$$= a_1 + \sqrt{x^2 + a_1(a_1 - 2a_2) - 2a_1(x - a_2)}$$

$$= a_1 + \sqrt{x^2 + a_1(a_1 - 2a_2) - 2a_1\sqrt{(x-a_2)^2}}$$

By continued recursion., we get

$$x = a_1 + \sqrt{x^2 + a_1(a_1 - 2a_2) - 2a_1\sqrt{x^2 + a_2(a_2 - 2a_3) - 2a_2\sqrt{x^2 + a_3(a_3 - 2a_4) - \ldots\ldots}}}$$ and so on.

Corollary:

When $a = a_1 = 2a_2 = 2^2a_3 = 2^3a_4 = \ldots\ldots\ldots\ldots\ldots\ldots = 2^{n-1}a_n \ldots\ \ldots\ldots$

We have $x = a + \sqrt{x^2 - 2a\sqrt{x^2 - a\sqrt{x^2 - a\sqrt{x^2 - a\ \ldots\ldots}}}}$ and so on.

---@@@@---

Problem 26: § SR Type Nested square root problems not from NBSR

1. $6 = \sqrt{36}$

$= \sqrt{6 + 5 \text{ x } 6}$

$= \sqrt{6 + 5\sqrt{36}} = \sqrt{6+5\sqrt{6+5\sqrt{6+5\sqrt{6+5\sqrt{6\ldots\ldots\ldots\ldots}}}}}$ and so on.

$6 = \sqrt{36}$

$= \sqrt{5 + 5^2 + \sqrt{36}}$

$= \sqrt{5+5^2+\sqrt{5+5^2+\sqrt{36}}}$

$= \sqrt{5+5^2+\sqrt{5+5^2+\sqrt{5+5^2+\sqrt{36}}}}$

$= \sqrt{5+5^2+\sqrt{5+5^2+\sqrt{5+5^2+\sqrt{5+5^2+\sqrt{36\ldots\ldots\ldots\ldots}}}}}$ and so on.

2. $6 = \sqrt{36}$

$= \sqrt{4 + 32}$

$= \sqrt{4 + 4\sqrt{64}}$

$= \sqrt{4 + 4\sqrt{4 + 4X15}}$

$= \sqrt{4+4\sqrt{4+4\sqrt{5+20X11}}}$

$= \sqrt{4+4\sqrt{4+4\sqrt{5+20\sqrt{121}}}}$

$= \sqrt{4+4\sqrt{4+4\sqrt{5+20\sqrt{5+4\sqrt{841}}}}}$ [$\because$121=5+116 =5+4 x 29=5+ $4\sqrt{841}$]

$= \sqrt{4+4\sqrt{4+4\sqrt{5+20\sqrt{5+4\sqrt{5+4\sqrt{2092+\ldots\ldots\ldots}}}}}}$ and so on.

7 $= \sqrt{49}$

$= \sqrt{7 + 42}$

$= \sqrt{7+6\sqrt{49}}$ [$42 = 6 \times 7 = 6\sqrt{49}$]

$= \sqrt{7+6\sqrt{7+42}}$

$= \sqrt{7+6\sqrt{7+6\sqrt{49}}}$

$= \sqrt{7+6\sqrt{7+6\sqrt{7+6\sqrt{7+6\sqrt{7+6\sqrt{49+\cdots\cdots\cdots}}}}}}$ and so on

7 $= \sqrt{49}$

$= \sqrt{10+39}$

$= \sqrt{10+3\sqrt{169}}$ [$\because 39 = 3 \times 13 = 3\sqrt{169}$]

$= \sqrt{10+3\sqrt{13+12\sqrt{169}}}$ [$\because 169 = 13 + 12 \times 13 = 13 + 12\sqrt{169}$]

$= \sqrt{10+3\sqrt{13+12\sqrt{13+12\sqrt{13+12\sqrt{13+12\sqrt{169+\cdots\cdots\cdots}}}}}}$ and so on.

13 $= \sqrt{169}$

$= \sqrt{13+12X13}$

$= \sqrt{13+12\sqrt{169}}$

$= \sqrt{13+12\sqrt{13+12\sqrt{13+12\sqrt{13+12\sqrt{13+12\sqrt{169+\cdots\cdots\cdots}}}}}}$ and so on.

--- -----@@@@--

------------------ (To be continued ……)

AUTHOR'S BIOGRAPHY

Prof. N. Ch. Pattabhi Ramacharyulu: He is a retired professor in Department of Mathematics & Humanities, National Institute of Technology, Warangal. He is a stalwart of Mathematics. His yeoman services as a lecturer, professor, Professor Emeritus and Deputy Director enriched the knowledge of thousands of students. He has nearly 42 PhDs and plenty number of M.phils to his credit. His papers more than 200 were published in various esteemed reputable International Journals. He is a Member of Various Professional Bodies. He published four books on Mathematics. He received so many prestigious awards and rewards.