



## Fixed Point Theorem on Complete Metric Space

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**Abstract:** *The aim of this paper we prove a simple result on complete metric space.*

**Keywords:** *Fixed-point, Complete metric Space*

### 1. INTRODUCTION

In many branches of science, economics, computer science, engineering and the development of nonlinear analysis, the fixed-point theory is one of the most important tools.

### 2. PRELIMINARIES

**Definition 2.1 :** - A Sequence  $\{x_n\}$  in Complete metric space

$(X, d)$  is called Cauchy sequence if given  $\epsilon > 0$  ,  $\exists n_0 \in \mathbb{N}$  such that  $\forall m, n \geq n_0$

$$d(x_m, x_n) < \epsilon \quad \text{or} \quad d(x_m, x_n) < \epsilon$$

i.e.  $\min \{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$

**Definition 2.2 :** - A Sequence  $\{x_n\}$  in Complete metric space converges to  $x$  if

$$\lim_{n \rightarrow \infty} d(x_n, x) = \lim_{n \rightarrow \infty} d(x, x_n) = 0 \quad \text{and } x \text{ is called limit of } \{x_n\} .$$

**Theorem 1:** Let  $F$  and  $G$  be mappings of a complete metric space  $(X, d)$  into  $B(X)$  Satisfying the inequality

$$\delta(Fx, Gy) \leq c. \max \{ \delta(x, Fx), \delta(y, Gy), d(x, y) \}$$

$\forall x, y \in X$  Where  $0 \leq c < 1$  then  $F$  and  $G$  have a common unique fixed point.

Inspired by above theorem we prove following theorem. (in main result)

### 3. MAIN RESULT

**Theorem 3.1** Let  $(X, d)$  be a complete metric space. Let  $S$  is a continuous mapping  $S: X \rightarrow X$  such that

$$d^2(Sx, Sy) \leq c. \max \{ d(x, y). d(x, Sx), d(x, y). d(y, Sy), d(x, y). d(x, Sy), d(x, Sx). d(y, Sy), d(x, Sx). d(x, Sy), d(y, Sy). d(x, Sy) \}$$

Where  $c \in (0,1)$  and  $x, y \in X$  and  $s \geq 1$  Then  $S$  has a unique fixed-point.

Proof: Let  $x_0 \in X$  be any arbitrary in  $X$  and  $\{x_n\}_{n=1}^{\infty}$  be a sequence in  $X$ . Defined by the recursion. Let

$$S(x_0) = x_1, \quad S(x_1) = x_2, \quad \text{in general.}$$

$$S(x_{n-1}) = x_n, \quad S(x_{n+1}) = x_{n+1} \quad , n = 0, 1, 2, 3 \dots \dots \dots$$

$$\begin{aligned} d^2(x_{n+1}, x_n) &= d^2(Sx_n, Sx_{n-1}) \\ &\leq c. \max \{ d(x_n, x_{n-1}). d(x_n, Sx_n), d(x_n, x_{n-1}). d(x_{n-1}, Sx_{n-1}), d(x_n, x_{n-1}). d(x_n, Sx_{n-1}), \\ &\quad d(x_n, Sx_n). d(x_{n-1}, Sx_{n-1}), d(x_n, Sx_n). d(x_n, Sx_{n-1}), d(x_{n-1}, Sx_{n-1}). d(x_n, Sx_{n-1}) \} \end{aligned}$$

$$\begin{aligned}
 & d^2(x_{n+1}, x_n) \\
 & \leq c \cdot \max\{ d(x_n, x_{n-1}) \cdot d(x_n, x_{n+1}), d(x_n, x_{n-1}) \cdot d(x_{n-1}, x_n), d(x_n, x_{n-1}) \cdot d(x_n, x_n), \\
 & \quad d(x_n, x_{n+1}) \cdot d(x_{n-1}, x_n), d(x_n, x_{n+1}) \cdot d(x_n, x_n), d(x_{n-1}, x_n) \cdot d(x_n, x_n) \} \\
 & d^2(x_{n+1}, x_n) \leq c \cdot \max\{ d(x_n, x_{n-1}) \cdot d(x_n, x_{n+1}), d(x_n, x_{n-1}) \cdot d(x_{n-1}, x_n), d(x_n, x_{n-1}) \cdot 0, \\
 & \quad d(x_n, x_{n+1}) \cdot d(x_{n-1}, x_n), d(x_n, x_{n+1}) \cdot 0, d(x_{n-1}, x_n) \cdot 0 \} \\
 & d^2(x_{n+1}, x_n) \leq c \cdot \max\{ d(x_n, x_{n-1}) \cdot d(x_n, x_{n+1}), d(x_n, x_{n-1}) \cdot d(x_{n-1}, x_n), \\
 & \quad d(x_n, x_{n+1}) \cdot d(x_{n-1}, x_n), \} \\
 & d^2(x_{n+1}, x_n) \leq c \cdot \max\{ d(x_n, x_{n-1}) \cdot d(x_n, x_{n+1}), d^2(x_n, x_{n-1}), \} \\
 & d^2(x_{n+1}, x_n) \leq c \cdot M_1
 \end{aligned}$$

Where  $M_1 = \max\{ d(x_n, x_{n-1}) \cdot d(x_n, x_{n+1}), d^2(x_n, x_{n-1}), \}$

Now two cases arises

Case I: If suppose that  $M_1 = d(x_n, x_{n-1}) \cdot d(x_n, x_{n+1})$

$$d^2(x_{n+1}, x_n) \leq c \cdot d(x_n, x_{n-1}) \cdot d(x_n, x_{n+1})$$

$$d(x_{n+1}, x_n) \leq c \cdot d(x_n, x_{n-1})$$

Where  $c = k \leq 1$

$$d(x_{n+1}, x_n) \leq k \cdot d(x_n, x_{n-1})$$

$$d(x_{n+1}, x_n) \leq k^2 \cdot d(x_{n-1}, x_{n-2})$$

Continuing this process

$$d(x_{n+1}, x_n) \leq k^n \cdot d(x_1, x_0)$$

Case II: If suppose that  $M_1 = d^2(x_n, x_{n-1})$

$$d^2(x_{n+1}, x_n) \leq c \cdot d^2(x_n, x_{n-1})$$

$$d(x_{n+1}, x_n) \leq \sqrt{c} \cdot d(x_n, x_{n-1})$$

Where  $h \leq \sqrt{c}$

$$d(x_{n+1}, x_n) \leq h \cdot d(x_n, x_{n-1})$$

$$d(x_{n+1}, x_n) \leq h^2 \cdot d(x_{n-1}, x_{n-2})$$

Continuing this process

$$d(x_{n+1}, x_n) \leq h^{n+1} \cdot d(x_1, x_0)$$

It follows that

$$d(x_n, x_{n+r}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+r-1}, x_{n+r})$$

$$\leq (h^n + \dots + h^{n+r-1}) \cdot d(x_0, x_1)$$

$$\leq \frac{h^n}{1-h} \cdot d(x_0, x_1)$$

Since  $h < 1$  and  $\epsilon > 0, \exists n_0 \in \mathbb{N}$

Such that  $d(x_m, x_n) \leq \epsilon, \forall m, n \geq n_0$

It follows that  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy Sequence converges to some  $x, x \in X$ , as  $X$  is complete.

Now

$$d(x, x_n) \leq d(x, x_m) + d(x_m, x_n)$$

$$\leq d(x, x_m) + \epsilon, \forall m, n \geq n_0$$

Let  $m \rightarrow \infty$

$$d(x, x_n) \leq \epsilon, \forall n \geq n_0$$

Thus

$$\begin{aligned} & d^2(Sx, Sx_{n-1}) \\ & \leq c \cdot \max\{d(x, x_{n-1}) \cdot d(x, Sx_n), d(x, x_{n-1}) \cdot d(x_{n-1}, Sx_{n-1}), d(x, x_{n-1}) \cdot d(x, Sx_{n-1}), \\ & \quad d(x, Sx) \cdot d(x_{n-1}, Sx_{n-1}), d(x, Sx) \cdot d(x, Sx_{n-1}), d(x_{n-1}, Sx_{n-1}) \cdot d(x, Sx_{n-1})\} \\ & d^2(Sx, Sx_{n-1}) \\ & \leq c \cdot \max\{d(x, x_{n-1}) \cdot d(x, Sx), d(x, x_{n-1}) \cdot d(x_{n-1}, x_n), d(x, x_{n-1}) \cdot d(x, x_n), \\ & \quad d(x, Sx) \cdot d(x_{n-1}, x_n), d(x, Sx) \cdot d(x, x_n), d(x_{n-1}, x_n) \cdot d(x, x_n)\} \end{aligned}$$

Taking  $n \rightarrow \infty$

$$d^2(Sx, Sx_{n-1}) = d^2(Sx, x_n) = d^2(Sx, x) \leq c \cdot \{0\} = 0$$

$$d^2(Sx, x) = 0$$

$$d(Sx, x) = 0$$

$$Sx = x$$

#### 4. UNIQUENESS OF FIXED-POINT

Assume that  $y$  is another fixed – point of  $X$ . Then we have

$$Sy = y \quad \text{and}$$

$$d^2(x, y) = d^2(Sx, Sy) \leq c \cdot \max\{d(x, y) \cdot d(x, Sx), d(x, y) \cdot d(y, Sy), d(x, y) \cdot d(x, Sy), \\ d(x, Sx) \cdot d(y, Sy), d(x, Sx) \cdot d(x, Sy), d(y, Sy) \cdot d(x, Sy)\}$$

$$d^2(x, y) \leq c \cdot \max\{d(x, y) \cdot d(x, x), d(x, y) \cdot d(y, y), d(x, y) \cdot d(x, y), \\ d(x, x) \cdot d(y, y), d(x, x) \cdot d(x, y), d(y, y) \cdot d(x, y)\}$$

$$d^2(x, y) \leq c \cdot \max\{0, 0, d(x, y) \cdot d(x, y), 0, 0, 0\}$$

$$d^2(x, y) \leq c \cdot d(x, y) \cdot d(x, y),$$

$$d(x, y) \leq c \cdot d(x, y)$$

This is contradiction . Therefore

$$x = y$$

This completes the proof . Hence  $x$  is the unique fixed – point of  $S$ .

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