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Generalised Lattice Metrized Spaces (gl-metrized spaces)

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Abstract: *In this paper initially introduced the concept generalised lattice betweenness (gl-betweenness) relation in generalised lattice and observed transitivity properties. Later introduced the concept generalised lattice metrized space (gl-metrized space), imagined triangles, sides of the triangles and observed their properties*

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1. Introduction

Mellacheruvu Krishna Murty and U. MadanaSwamy (Professors of Andhra University)[6] introduced the concept of generalised lattice. The author P.R.Kishore [2,3,4] developed the theory of generalised lattices that can play an intermediate role between the theories of lattices and posets. The concept of Lattice metrized space (L-metrized space) is known from Leo Lapidus [5,7] and later in [8] the concept of Brouwerian generalised lattice introduced and developed by the author P.R.Kishore. In this paper section 2 contains some preliminary concepts that are from the references. In section 3 introduced two kinds of transitivity properties t_1, t_2 and generalised lattice betweenness (gl-betweeness) relation in a generalised lattice. Proved that the gl-betweeness relation satisfies the transitivity t_1. In section 4 introduced the concept generalised lattice metrized space (gl-metrized space), imagined triangles, sides of the triangles and observed their properties. Finally introduced a relation P-linear (Pl) in a gl-metrized space and proved that it satisfies the transitivity t_2.

2. Preliminaries

This section contains some preliminaries from the references those are useful in the next sections.

[Murty [6]] For any finite subset A of a poset P, define L(A) = {x ϵ P | x ≤ a for all a ϵ A} and U(A) = {x ϵ P | a ≤ x for all a ϵ A}. Then the sets L(P) = {L(A) | A is a finite subset of P} and U(P) = {U(A) | A is a finite subset of P} are semi lattices under set inclusion.

Definition 2.1 [Murty [6]] Let (P, ≤) be a poset. P is said to be a generalised meet semilattice if for every non empty finite subset A of P, there exist a non-empty finite subset B of P such that, x ϵ L(A) if and only if x ≤ b for some b ϵ B. P is said to be a generalised join semilattice if for every non empty finite subset A of P, there exist a non-empty finite subset B of P such that, x ϵ U(A) if and only if b ≤ x for some b ϵ B. P is said to be a generalised lattice if it is both generalised meet and join semilattice.

[Murty [6]] It is observed that if P is a generalised meet (join) semilattice, then for any L(A) ϵL(P)(U(A) ϵU(P)) there exists a unique finite subset B of P such that $L(A)=\cup_{b \epsilon B} L(b)$ $(U(A) = \cup_{b \epsilon B} U(b))$ and the elements of B are mutually incomparable and the set is denoted by ML(A) (mu(A)).If a posetP is a generalised lattice then (L(P), ⊆) and (U(P), ⊆) are lattices.

3. Generalised Lattice Betweenness (GL-Betweenness)

Definition 3.1 Let P be a generalised lattice and $\theta \subseteq P \times P \times P = P^3$. Then θ is said to have the property of transitivity t_1 if for a,b,c,x ϵ P; (a,b,c)\ ϵθ and (a,x,b) ϵ θ implies (x,b,c) ϵ θ.

Definition 3.2 Let P be a generalised lattice and $\theta \subseteq P^3$. Then θ is said to have the property of transitivity t_2 if for a,b,c,x ϵ P; (a,b,c) ϵθ and (a,x,b) ϵθ implies (a,x,c) ϵ θ.

Definition 3.3 Let P be a generalised lattice and a,b,c ϵ P. Then b is said to be gl-between a and c in P if ML(mu(ML{a, b} ∪ ML{b, c})) = {b} = mu(ML(mu{a,b} ∪ mu{b,c})), denoted by (a,b,c)glb.

Note: Let P be a generalised lattice. Then glb = {(a,b,c) ϵ P^3| (a,b,c)glb} = \{(a,b,c) ϵ P^3| ML(mu(ML{a, b} ∪ ML{b, c})) = {b} = mu(ML(mu{a,b} ∪ mu{b,c})) is a 3-relation on P.

Theorem 3.4 Let P be a generalised lattice. Then the 3-relation glb on P has transitivity t_1.

Proof: Let a,b,c,x ϵ P and suppose (a,b,c), (a,x,b) ϵ glb. Then ML(mu(ML{a, b} ∪ ML{b, c})) = {b} = mu(ML(mu{a,b} ∪ mu{b,c})) and ML(mu{a, x} ∪ mu{x, b}) = {x} = mu(ML(mu{a,x} ∪ mu{x,b})). Consider L(x)∧ (L(a)∧ L(b)) = (L(a) ∨ L(x))∧ (L(x)∨ L(b)) ∧ (L(a) ∧ L(b)) = L(a) ∧ L(b). To show that (x,b,c) ϵ glb: To show that ML(mu(ML{x,b} ∪ ML{b,c})) = {b}: Consider L(mu(ML{x,b} ∪ ML{b,c})) = (L(x) ∧ L(b)) ∨ (L(b) ∧ L(c)) = (L(x) ∧ ((L(a) ∧ L(b)) ∨ (L(b) ∧ L(c))) ∨ (L(b) ∧ L(c)) ≥ (L(x) ∧ L(a) ∧ L(b)) ∨ (L(x) ∧ L(b) ∧ L(c)) ∨ (L(b) ∧ L(c)) = (L(a) ∧ L(b)) ∨ (L(b) ∧ L(c)) = L(b). Again consider L(mu(ML{x,b} ∪ ML{b,c})) = (L(x) ∧ L(b)) ∨ (L(b) ∧ L(c)) ≤ L(b) ∨ L(b) = L(b). Therefore L(mu(ML{x,b} ∪ ML{b,c})) = L(b). Therefore ML(mu(ML{x,b} ∪ ML{b,c})) = {b}. To show that ML(mu{x,b}∪ mu{b,c})) = {b}: We know that {b} ⊆ {x, b} and {b} ⊆ {b, c}. This implies U({x, b}) ⊆ U({b}) and U({b, c}) ⊆ U({b}). This implies L(b) = L(U({b})) ⊆ L(U({x, b})) and L(b) = L(U({b})) ⊆ L(U({b, c})). Then L(b) ⊆ L(U({x, b})) ∩ L(U({b, c})) = L(mu{x,b} ∪ mu{b,c})). This implies U(ML(mu{x,b} ∪ mu{b,c}))) ⊆ U(L(b)) = U(b). Consider U(x) ∧ (U(a) ∧ U(b)) = (U(a) ∨U(x)) ∧ (U(x) ∨ U(b)) ∧ (U(a) ∧ U(b)) = U(a) ∧ U(b). Consider U(ML(mu{x,b} ∪ mu{b,c})) = U({x,b}) ∨ U({b,c}) = (U(x) ∧ U(b)) ∨ (U(b) ∧U(c)) = (U(x) ∧ ((U(a) ∧ U(b)) ∨ (U(b) ∧ U(c))) ∨ (U(b) ∧ U(c)) ≥ (U(x) ∧ U(a) ∧ U(b)) ∨ (U(x) ∧ U(b) ∧ U(c)) ∨ (U(b) ∧ U(c)) = (U(a) ∧ U(b)) ∨ (U(b) ∧ U(c)) = U(b). Therefore mu(ML(mu{x,b} ∪ mu{b,c})) = {b}. Therefore (x,b,c) ϵ glb.□

4. GENERALISED LATTICE METRIZED SPACES (GL-METRIZED SPACES)

Definition 4.1 Let S be a set and P be a generalised lattice having least element 0. If there exists a map d: S x S → P such that (i) d(a,b) ≥ 0 and d(a,b) = 0 ⟹ a = b (ii) d(a,b) = d(b,a) (iii) d(a,c) ϵ L(mu(d(a,b), d(b,c))), then the ordered pair (S,d) is called a generalised lattice metrized space (gl-metrized space). We denote d(a,b) by a ∗ b.

Definition 4.2 Let S be a gl-metrized space and a, b, c ϵ S. Then one can imagin a triangle in S with vertices a, b, c and sides a ∗ b, b ∗ c and c ∗ a, called a P-triangle in S, denoted by Δ_P (a, b, c).

Theorem 4.3 Let S be a gl-metrized space and Δ_P (a, b, c) be a P-triangle in S. Let x=a ∗ b, y=b ∗ c and z=c ∗ a. Then L(mu{x, y}) = L(mu{y, z}) = L(mu{z, x}).

Proof: Clearly x, y, z ϵ P. Since P is a generalised lattice, by Murty [6] we have L(P) is a lattice. Since S is a gl-metrized space by definition 4.1, we have z = c ∗ a = a ∗ c ϵL(mu{a ∗ b, b ∗ c}) = L(mu{x, y}). This implies L(z)⊆ L(x)∨ L(y). Similarly we get L(y)⊆L(z)∨ L(x) and L(x)⊆ L(y)∨ L(z). Then we have L(x)∨ L(y)⊆ L(y)∨ L(z)⊆ L(x)∨ L(y). This means L(x)∨ L(y) = L(y)∨ L(z). Similarly we get L(y)∨ L(z) = L(z)∨ L(x) and L(z)∨ L(x) = L(x)∨ L(y). Then L(x)∨ L(y) = L(y) ∨ L(z) = L(z) ∨ L(x). Therefore L(mu{x, y}) = L(mu{y, z}) = L(mu{z, x}).□

Definition 4.4 Let S be a gl-metrized space and a, b, c ϵ S. Then b is said to be metrically between a and c if L(mu{a ∗ b, b ∗ c}) = L(a ∗ c).

Definition 4.5 Let S be a gl-metrized space and a, b, c ϵ S. If b is said to be metrically between a and c then we say that a, b, c are P-linear, denoted by (a, b, c)Pl.

Note: Let S be a gl-metrized space. Then by definitions 4.4 and 4.5 we have Pl = {(a,b,c) ϵ S x S x S | (a, b, c)Pl} = {(a,b,c) ϵ S^3| b is said to be metrically between a and c} = {(a,b,c) ϵ S^3| L(mu{a ∗ b, b ∗ c}) = L(a ∗ c)} is a 3-relation on S.

Theorem 4.6 Let S be a gl-metrized space and a, b, c ϵ S. Then (a, b, c) ϵ Pl if and only if (c, b, a) ϵ Pl.

Proof: (a, b, c) ϵ Pl if and only if L(mu{a ∗ b, b ∗ c}) = L(a ∗ c) if and only if L(mu{c ∗ b, b ∗ a}) = L(c ∗ a) (by definition 4.1) if and only if (c, b, a) ϵ Pl. □

Definition 4.7 Let S be a gl-metrized space and a, b, c ϵ S. Then the triple of elements (a, b, c) is said to satisfy P-line segment property (Pls property) if (a, b, c) ϵ Pl and (a, c, b) ϵ Pl if and only if b= c.

Theorem 4.8 Let S be a gl-metrized space and a, b, c ϵ S. If a, b, c are vertices of an isosceles triangle then (a, b, c) ϵ Pl implies (a, c, b) ϵ Pl.

Proof: Suppose a, b, c are vertices of an isosceles triangle and say that a ∗ b = a ∗ c. Suppose (a, b, c) ϵ Pl. Then we have L(mu{a ∗ b, b ∗ c}) = L(a ∗c). Since L(P) is a lattice, we get L(a ∗ b)∨L(b ∗ c) = L(a ∗ c) = L(a ∗ b). To show that (a, c, b) ϵ Pl: By definition 4.1 we have a ∗ b ϵ L(mu{a ∗ c, c ∗ b}). Since L(P) is a lattice, we get L(a ∗ b) ⊆ L(a ∗ c)∨ L(c ∗ b) = L(a ∗ c)∨ L(b ∗ c) = L(a ∗ b)∨ L(b ∗ c) = L(a ∗ b). Therefore L(mu{a ∗ c, c ∗ b}) = L(a ∗ c)∨L(c ∗ b) = L(a ∗ b). That is (a, c, b) ϵ Pl. □

Theorem 4.9 Let S be a gl-metrized space and a, b, c ϵ S. Suppose (a, b, c) ϵ Pl. Then (a, b, c) not satisfies P-line segment property (Pls property) if and only if a, b, c are vertices of an isosceles triangle.

Proof: Suppose a, b, c are vertices of an isosceles triangle. That is a≠ b ≠ c. Given that (a, b, c) ϵ Pl. Then by theorem 4.8 we get (a, c, b) ϵ Pl. Since b≠ c, by definition 4.7, we can say that (a, b, c) not satisfies Pls property. Conversely suppose (a, b, c) not satisfies Pls property. To show that a, b, c are vertices of an isosceles triangle: If a = c then a ∗ b = c ∗ b = b ∗ c and therefore Δ_P(a,b,c) is an isosceles triangle. Suppose a≠ c. Then Δ_P(a,b,c) is a P-triangle in S with a≠ c. Case(i): Suppose (a,c,b) ϵ Pl. Then we have (a, b, c) ϵ Pl and (a,c,b) ϵ Pl. Now by theorem 4.3 and definition 4.4 we have L(a ∗ c) = L(mu{a ∗ b, b ∗ c}) = L(mu{a ∗ c, c ∗ b}) = L(a ∗ b). Therefore a ∗ c = a ∗ b.Case(ii): Suppose (a,c,b) ϵ S^3 - Pl. Then since (a, b, c) not satisfies Pls property, by definition 4.7 we get b=c. Therefore a ∗ c = a ∗ b. Hence by both the cases we can say that Δ_P (a,b,c) is an isosceles triangle. □

Theorem 4.10 Let S be a gl-metrized space. Then the 3-relation Pl on S satisfies the property of transitivity t_2.

Proof: Let a,b,c,x ϵ S. Suppose (a,b,c) ϵ Pl and (a,x,b) ϵ Pl. To show that (a,x,c) ϵ Pl: By note after definition 4.5 we get L(a ∗ b)∨ L(b ∗ c) = L(mu{a ∗ b, b ∗ c}) = L(a ∗ c) and L(a ∗ x)∨ L(x ∗ b) = L(mu{a ∗ x, x ∗ b}) = L(a ∗ b). Then L(a ∗ x)∨ L(x ∗ b)∨ L(b ∗ c) = L(a ∗ b)∨ L(b ∗ c) = L(a ∗ c). By definition 4.1 we have x ∗ c ϵ L(mu({x ∗ b, b ∗ c})) and a ∗ c ϵ L(mu({a ∗ x, x ∗ c})). Then L(a ∗ c)⊆ L(a ∗ x)∨ L(x ∗ c)⊆ L(a ∗ x)∨ L(x ∗ b)∨ L(b ∗ c) = L(a ∗ c). That is L(mu{a ∗ x, x ∗ c}) = L(a ∗ x)∨ L(x ∗ c) = L(a ∗ c). Therefore (a,x,c) ϵ Pl. Therefore Pl satisfies the property of transitivity t_2. □

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