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Representation of Pre A* - Algebra by a Partially Order

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Abstract: *This manuscript is a study on Pre-A*-algebra A in view of it is like a partially ordered set. Using a binary operation in Pre-A*-algebra, an observation is made on Pre A*-Algebra as a partially ordered set with respect to binary operation $\wedge$ and obtained corresponding results. It is also make available an equivalent condition for a Pre A*-algebra become a Boolean algebra.*

Keywords: *A*-algebra, Pre-A*-algebra, Boolean algebra, Partially ordered set, Ada, Homomorphism.*

AMS subject classification (2000): 06E05, 06E25, 06E99, 06B10

1. Introduction

In a draft manuscript entitled "The Equational theory of Disjoint Alternatives", E. G. Manes (1989) introduced the concept of Ada (Algebra of disjoint alternatives) $(A, \wedge, \vee, (\text{-})', (\text{-})_\pi, 0, 1, 2)$ which is however differs from the definition of the Ada of E. G. Manes (1993) later paper entitled "Adas and the equational theory of if-then-else". While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras and the later concept is based on C-algebras A ($\wedge$, $\vee$ ‘) introduced by Fernando Guzman and Craig C. Squir (1990). P. Koteswara Rao (1994) first introduced the concept of A*-algebra $(A, \wedge, \vee, *, (-)^{\sim} (\text{-})_\pi, 0, 1, 2)$ not only studied the equivalence with Ada, C-algebra, Ada's connection with 3-Ring, Stone type representation but also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra.

J.Venkateswara Rao (2000) introduced the concept Pre A*-algebra $(A, \wedge, \vee, (-)^{\sim})$ analogous to C-algebra as a reduct of A*- algebra. Venkateswara Rao.J, Praroopa.Y (2006) made a structural study on Boolean algebras and Pre A*-Algebras.

Boolean algebra depends on two element logic. C-algebra, Ada, A*- algebra and our Pre A*-algebra are regular extensions of Boolean logic to 3 truth values, where the third truth value stands for an undefined truth value. The Pre A*- algebra structure is denoted by $(A, \wedge, \vee, (-)^{\sim})$ where A is non-empty set, $\wedge, \vee$ are binary operations and $(-)^{\sim}$ is a unary operation.

In this paper we define a relation $\leq$ on Pre A*-algebra with respect to the binary operation $\wedge$, we discuss the properties of a Pre A*-algebra like a poset. We find the necessary conditions for a poset to become a lattice. We also present a equivalent condition for a Pre A*-algebra become a Boolean algebra. For any $a \in A$ define $A_a = \{x \in A \,/\, a \wedge x = x\}$ and $x^a = a \wedge x^{\sim}$ then (A_a , $\wedge, \vee, {}^a$)

is a Pre A*-algebra. We also define a mapping $\alpha_{a,b}$ from A_b to A_a by $\alpha_{a,b}$ (x) = a$\wedge$x for all x$\in A_b$ is a homomorphism of Pre A*-algebras.

PRELIMINARIES

1.1. Definition: The relation R on a set A is called a partial order on A when R(≤) is reflexive, anti-symmetric, and transitive. Under these conditions, the set A is called a partially ordered set or a poset. Frequently we write (A, R) or (A, ≤) to denote that A is partially ordered by the relation R(≤). Since the relation ≤ on the set of real numbers is the prototype of a partial order it is common to write ≤ to represent an arbitrary partial order can be described as follows:

1. For all a ∈ A, a ≤ a (reflexive)
2. For all a, b ∈ A, a ≤ b, b ≤ a, then a = b (anti symmetry)
3. For all a, b, c ∈ A, a ≤ b and b ≤ c, then a ≤ c (transitivity)

Two elements a and b in A are said to be comparable under ≤ if either a ≤ b or b ≤ a; otherwise they are incomparable. If every pair of elements of A are comparable, then we say that the partially ordered set is totally ordered.

1.2. Definition: An algebra $(A, \wedge, \vee, (-)^{\sim})$ where A is a non-empty set with 1, $\wedge$, $\vee$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying

(a) $x^{\sim\sim} = x \quad \forall x \in A$

(b) $x \wedge x = x, \quad \forall x \in A$

(c) $x \wedge y = y \wedge x, \quad \forall x, y \in A$

(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim} \quad \forall x, y \in A$

(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad \forall x, y, z \in A$

(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad \forall x, y, z \in A$

(g) $x \wedge y = x \wedge (x^{\sim} \vee y), \quad \forall x, y \in A$ is called a Pre A*-algebra.

1.1. Example: **3** = {0, 1, 2} with operations $\wedge$, $\vee$, $(-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

x	$x^{\sim}$
0	1
1	0
2	2

1.1. Note: The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$ (b) 1 $\wedge$ x = x for all x ∈ **3**

(c) 0 $\vee$ x = x for all x ∈ **3** (d) 2 $\wedge$ x = 2 $\vee$ x = 2 for all x ∈ **3**.

1.2. Example: **2** = {0, 1} with operations$\wedge$,$\vee$ $(-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

x	$x^{\sim}$
0	1
1	0

1.2. Note :(i) $(2,\vee,\wedge,(-)^{\sim})$ is a Boolean algebra. So every Boolean algebra is a Pre A* algebra.

(ii) The identities 1.2(a) and 1.2(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.2(b) to 1.2(g).

1.3. Definition: Let A be a Pre A*-algebra. An element x $\in$A is called a central element of A if $x\vee x^{\sim}=1$ and the set {x $\in$ A/ $x\vee x^{\sim}=1$} of all central elements of A is called the centre of A and it is denoted by B (A).

1.1. Theorem: [Satyanarayana.A, (2012)] Let A be a Pre A*-algebra with 1, then B (A) is a Boolean algebra with the induced operations $\wedge$, $\vee$, $(-)^{\sim}$

1.1. Lemma: [Satyanarayana.A, (2012)] Every Pre A*-algebra with 1 satisfies the following laws

(a) $x\vee 1 = x\vee x^{\sim}$ (b) $x\wedge 0 = x\wedge x^{\sim}$

1.2. Lemma: [Satyanarayana.A, (2012)] Every Pre A*-algebra with 1 satisfies the following laws.

(a) $x\wedge(x^{\sim}\vee x) = x\vee(x^{\sim}\wedge x) = x$

(b) $(x\vee x^{\sim})\wedge y = (x\wedge y)\vee(x^{\sim}\wedge y)$

(c) $(x\vee y)\wedge z = (x\wedge z)\vee(x^{\sim}\wedge y\wedge z)$

1. 4. Definition: Let $(A_1,\vee,\wedge,(-)^{\sim})$ and $(A_2,\vee,\wedge,(-)^{\sim})$ be a two Pre A*- algebras. A mapping $f: A_1 \to A_2$ is called a Pre A*-homomorphism if

(i) $f(a\wedge b) = f(a)\wedge f(b)$ (ii) $f(a\vee b) = f(a)\vee f(b)$ (iii) $f(a^{\sim}) = (f(a))^{\sim}$

The homomorphism $f: A_1 \to A_2$ is onto, then f is called epimorphism.

The homomorphism $f: A_1 \to A_2$ is one-one then f is called monomorphism

The homomorphism $f: A_1 \to A_2$ is one-one and onto then f is called an isomorphism, and A_1, A_2 are isomorphic, denoted in symbol $A_1 \cong A_2$.

2. Pre A*- Algebra as a Poset with Respect to Binary Operation $\wedge$

2. 1 Definition: Let A be a Pre A*-algebra. Define a relation ≤ on A by x ≤ y if and only if y $\wedge$ x = x $\wedge$ y = x.

2. 1 Lemma: If A is a Pre A*-algebra, then (A, ≤) is a poset.

Proof: Since x $\wedge$ x = x, x ≤ x for all x $\in$ A

Therefore ≤ is reflexive.

Suppose that x, y, z $\in$ A, x ≤ y and y ≤ z.

Then we have y $\wedge$ x = x $\wedge$ y = x and z $\wedge$ y = y $\wedge$ z = y.

Now x = x $\wedge$ y = x $\wedge$ y $\wedge$ z = x $\wedge$ z. $\Rightarrow$ x $\wedge$ z = z $\wedge$ x = x

Therefore, x ≤ z. This shows that ≤ is transitive.

Suppose that x, y $\in$ A, x ≤ y and y ≤ x $\Rightarrow$ y $\wedge$ x = x $\wedge$ y = x and y $\wedge$ x = x $\wedge$ y = y.

This shows that x = y. Therefore ≤ is anti-symmetric. Hence (A, ≤) is poset.

2. 1 Note: If A is a Pre A*-algebra with 1, 0, 2 then $x \leq 1 (x \wedge 1 = 1 \wedge x = x)$, for all $x \in A$ and $2 \leq x$ $(x \wedge 2 = 2 \wedge x = 2)$. This shows that 1 is the greatest element and 2 is the least element of the poset. The Hasse diagram of the poset (A, ≤) is given by

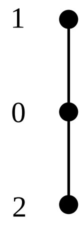

Diagram 2.1

We have $A \times A = \{a_1 = (1,1),\ a_2 = (1,0),\ a_3 = (1,2),\ a_4 = (0,1),\ a_5 = (0,0),\ a_6 = (0,2),\ a_7 = (2,1),\ a_8 = (2,0),\ a_9 = (2,2)\}$ is a Pre A*-algebra under point wise operation and $A \times A$ is having four central elements and remaining are non central elements, among that $a_9 = (2,2)$ is satisfying the property that $a_9^{\sim} = a_9$. The Hasse diagram is of the poset $(A \times A, \leq)$ given below

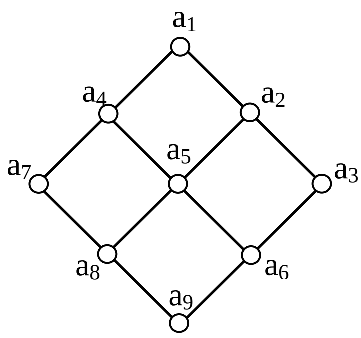

Diagram 2.2

Observe that, $x \leq a_1$, $x \wedge a_1 = a_1 \wedge x = x$ and $a_9 \leq x (x \wedge a_9 = a_9 \wedge x = a_9)$ for all $x \in A \times A$. This shows that a_1 is the greatest element and a_9 is the least element of $A \times A$.

We have $\mathbf{2} \times \mathbf{3} = \{a_1 = (1,1), a_2 = (0,0), a_3 = (1,0), a_4 = (0,1), a_5 = (2,2),$
$a_6 = (1,2)\}$ is a Pre A*-algebra under point wise operation having four central elements, two non-central elements and no element is satisfying the property that $a^{\sim} = a$.

The Hasse diagram for $(\mathbf{2} \times \mathbf{3}, \leq)$ as given below

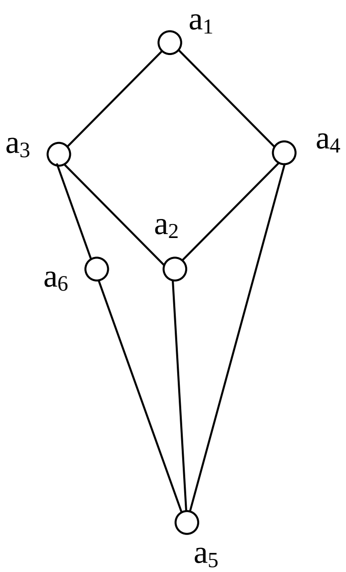

Diagram 2.3

Observe that, $x \leq a_1$, that is, $x \wedge a_1 = a_1 \wedge x = x$ and $a_5 \leq x$ ($x \wedge a_5 = a_5 \wedge x = a_5$) for all $x \in \mathbf{2} \times \mathbf{3}$. This shows that a_1 is the greatest element and a_5 is the least element of $\mathbf{2} \times \mathbf{3}$.

2. 1. Theorem: In the partially ordered set (A, ≤), for any $x \in A$, supremum of $\{x, x^{\sim}\} = x \vee x^{\sim}$ and infimum $\{x, x^{\sim}\} = x \wedge x^{\sim}$.

Proof: We have $(x \vee x^{\sim}) \wedge x = x$ and $x^{\sim} \wedge (x \vee x^{\sim}) = x^{\sim}$

Therefore $x \leq x \vee x^{\sim}$ and $x^{\sim} \leq x \vee x^{\sim}$

Hence $x \vee x^{\sim}$ is an upper bound of $\{x, x^{\sim}\}$

Suppose n is an upper bound of $\{x, x^{\sim}\}$

That is, $x \leq n$, $x^{\sim} \leq n \Rightarrow n \wedge x = x$, and $n \wedge x^{\sim} = x^{\sim}$

Now $n \wedge (x \vee x^{\sim}) = (n \wedge x) \vee (n \wedge x^{\sim}) = x \vee x^{\sim}$

This shows that $x \vee x^{\sim} \leq n$

Therefore $x \vee x^{\sim}$ is a least upper bound of $\{x, x^{\sim}\}$

This shows that supremum of $\{x, x^{\sim}\} = x \vee x^{\sim}$

Again we have $(x \wedge x^{\sim}) \wedge x = x \wedge x^{\sim}$ and $(x \wedge x^{\sim}) \wedge x^{\sim} = x \wedge x^{\sim}$

Therefore $x \wedge x^{\sim} \leq x$ and $x \wedge x^{\sim} \leq x^{\sim}$

Hence $x \wedge x^{\sim}$ is a lower bound of $\{x, x^{\sim}\}$

Suppose m is a lower bound of $\{x, x^{\sim}\}$

That is, $m \leq x$, $m \leq x^{\sim} \Rightarrow m \wedge x = m$, and $m \wedge x^{\sim} = m$

Now $m \wedge (x \wedge x^{\sim}) = (m \wedge x) \wedge x^{\sim} = m \wedge x^{\sim} = m$

This shows that $m \leq x \wedge x^{\sim}$

Therefore $x \wedge x^{\sim}$ is a greatest lower bound of $\{x, x^{\sim}\}$

This shows that infimum of $\{x, x^{\sim}\} = x \wedge x^{\sim}$

2. 2. Theorem: In a poset (A, ≤) with 1, for any $x, y \in A$, $\mathrm{Inf}\{x,y\} = x \wedge y$.

Proof: We have $(x \wedge y) \wedge x = x \wedge y$ and $(x \wedge y) \wedge y = x \wedge y$

Therefore $x \wedge y \leq x$ and $x \wedge y \leq y$.

Hence $x \wedge y$ is a lower bound of $\{x, y\}$

Suppose m is a lower bound of $\{x, y\}$

That is, $m \leq x$, $m \leq y \Rightarrow m \wedge x = m$ and $m \wedge y = m$

Now $m \wedge (x \wedge y) = (m \wedge x) \wedge y = m \wedge y = m$.

This shows that $m \leq x \wedge y$

Therefore $x \wedge y$ is a greatest lower bound of $\{x,y\}$

This shows that infimum of $\{x, y\} = x \wedge y$.

In general for a Pre A*-algebra with 1, $x \vee y$ need not be the l.u.b of $\{x, y\}$ in (A, ≤). For example $2 \vee x = 2 \wedge x = 2$, $\forall x \in A$ is not a least upper bound. However we have the following theorem.

2. 3. Theorem: In a poset (A, ≤) with 1, for any $x, y \in B(A)$, $\sup\{x, y\} = x \vee y$.

Proof: If $x, y \in B(A)$, then we have, $x \wedge (x \vee y) = x$ and $y \wedge (x \vee y) = y$

This shows that $x \leq x \vee y$ and $y \leq x \vee y$

Hence $x \vee y$ is an upper bound of $\{x,y\}$

Suppose z is an upper bound of {x,y}, then $z \wedge x = x$, $z \wedge y = y$

Now $z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y) = x \vee y$

Therefore, $x \vee y \leq z$.

Hence sup $\{x, y\} = x \vee y$.

2.4 Theorem: In the poset $(A, \leq)$, if $x,y \in B(A)$, then $x \vee y \leq x \vee x^{\sim}$.

Proof: $(x \vee x^{\sim}) \wedge (x \vee y) \quad = \{x \wedge (x \vee y)\} \vee \{x^{\sim} \wedge (x \vee y)\}$

$= x \vee (x^{\sim} \wedge y)$

$= x \vee y$

Therefore $x \vee y \leq x \vee x^{\sim}$

2.5. Theorem: In the poset $(A, \leq)$, if $x \leq y$, then for any $z \in A$,

(a) $z \wedge x \leq z \wedge y$

(b) $z \vee x \leq z \vee y$

Proof: If $x \leq y$, then $x \wedge y = x$

(a) $(z \wedge x) \wedge (z \wedge y) = \{(z \wedge x) \wedge z\} \wedge y = (z \wedge x) \wedge y = z \wedge x$.

Therefore $z \wedge x \leq z \wedge y$

(b) $(z \vee x) \wedge (z \vee y) = z \vee (x \wedge y) = z \vee x$

Therefore $z \vee x \leq z \vee y$

Now we are giving the following equivalent conditions for $x \leq y$.

2. 2. Lemma: In a Pre A*-algebra (i) $x \leq y \Leftrightarrow x \wedge (x^{\sim} \vee y) = (x^{\sim} \vee y) \wedge x = x$

(ii) $x \leq y \Leftrightarrow y \wedge (y^{\sim} \vee x) = (y^{\sim} \vee x) = (y^{\sim} \vee x) \wedge y = x$

Proof: (i) If $x \leq y \quad \Leftrightarrow x \wedge y = x$

$\Leftrightarrow x \wedge (x^{\sim} \vee y) = (x^{\sim} \vee y) \wedge x = x$

(ii) If $x \leq y \Leftrightarrow y \wedge x = x$

$\Leftrightarrow y \wedge (y^{\sim} \vee x) = (y^{\sim} \vee x) \wedge y = x$

Now we prove modular type results in the following lemma.

2.3 Lemma: In the poset $(A, \leq)$, if $x \leq y \Rightarrow x \vee (y \wedge z) = y \wedge (x \vee z)$.

Proof: Suppose $x \leq y$ then $y \wedge x = x$

Now $y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z) = x \vee (y \wedge z)$

If $x, y \in B(A)$ then by theorem 2. 3, sup $\{x, y\} = x \vee y$. In general $x \vee y$ need not be an upper bound of {x,y} in poset $(A, \leq)$. If $x \vee y$ is an upper bound of {x,y} in poset $(A, \leq)$, then A becomes Boolean algebra. Now we have the following theorem.

2.6. Theorem: If A is a Pre A*-algebra and $x \wedge (x \vee y) = x$ for all $x, y \in A$ then $(A, \leq)$ is a lattice.

Proof: By Theorem 2.2, we have every pair of elements have g.l.b and if $x \wedge (x \vee y) = x$ for all $x, y \in A$, then by theorem 2.3 we have every pair of elements have l.u.b. Hence $(A, \leq)$ is a lattice.

Now we present an equivalent condition for a Pre A*-algebra become a Boolean algebra.

2.7. Theorem: The following conditions are equivalent for any Pre A*-algebra $(A, \wedge, \vee, (-)^{\sim})$.

(1) A is a Boolean Algebra

(2) $x \leq x \vee y$ for all x, $y \in A$
(3) $y \leq x \vee y$ for all x, $y \in A$
(4) $x \vee y$ is an upper bound of {x, y} in (A, $\leq$) for all x, $y \in A$
(5) $x \vee y$ is an supremum of {x, y} in (A, $\leq$) for all x, $y \in A$
(6) $x \vee x^{\sim}$ is the greatest element in (A, $\leq$) for every $x \in A$

Proof: (1) $\Rightarrow$(2) Suppose A be a Boolean algebra

Now $x \wedge (x \vee y) = x$ (by absorption law)

Hence $x \leq x \vee y$.

(2) $\Rightarrow$(3) suppose $x \leq x \vee y$ then $x \wedge (x \vee y) = x$

Now $y \wedge (x \vee y) = y$ .Therefore $y \leq x \vee y$.

(3) $\Rightarrow$(4) Suppose that $y \leq x \vee y \Rightarrow y \wedge (x \vee y) = y$

Since $y \leq x \vee y$ then $x \vee y$ is upper bound of y

Now $x \wedge (x \vee y) = x$ (by supposition)

Therefore $x \leq x \vee y \Rightarrow x \vee y$ is upper bound of x

Hence $x \vee y$ is an upper bound of {x, y}.

(4) $\Rightarrow$(5) suppose $x \vee y$ is an upper bound of {x, y}

Suppose z is an upper bound of {x, y}, then $x \leq z$, $y \leq z$ that is $x \wedge z = x$, $y \wedge z = y$

Now $z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y) = x \vee y$

Therefore $x \vee y \leq z$. Hence sup{x, y} = $x \vee y$.

(5) $\Rightarrow$(6) suppose sup$\{x, y\} = x \vee y$ then $x, y \in B(A)$

Now sup$\{x \vee x^{\sim}, y\} = x \vee x^{\sim} \vee y = x \vee x^{\sim}$

$\Rightarrow y \leq x \vee x^{\sim}$

Therefore $x \vee x^{\sim}$ is the greatest element in (A, $\leq$).

(6) $\Rightarrow$(1) suppose $x \vee x^{\sim}$ is the greatest element in A then $y \leq x \vee x^{\sim}$

$\Rightarrow (x \vee x^{\sim}) \wedge y = y$

Now $y \vee (x \wedge y) = [(x \vee x^{\sim}) \wedge y] \vee (x \wedge y) = [(x \vee x^{\sim}) \vee x] \wedge y$

$= (x \vee x^{\sim}) \wedge y = y$ (by supposition)

Therefore absorption law holds hence A is a Boolean algebra.

2.8. Theorem: Let A be a pre A*-algebra if $x \wedge x^{\sim}$ is the least element in

(A, $\leq$) for every $x \in A$, then A is a Boolean algebra.

Proof: Suppose $x \wedge x^{\sim}$ is the least element in (A, $\leq$) then $x \wedge x^{\sim} \leq y$

$\Rightarrow (x \wedge x^{\sim}) \wedge y = x \wedge x^{\sim}$

Now $x \wedge (x \vee y) = [x \vee (x^{\sim} \wedge x)] \wedge (x \vee y)$

$= x \vee [(x \wedge x^{\sim}) \wedge y]$

$= x \vee (x \wedge x^{\sim})$ (by supposition)

$= x$

Therefore $x \wedge (x \vee y) = x$, absorption law holds.

Therefore A is a Boolean algebra.

2.9. Theorem: Let A be a Pre A*-algebra and $a \in A$. Let

$A_a = \{x \in A \ / \ a \wedge x = x\}$.Then A_a is closed under the operations $\wedge$ and $\vee$. Also for any $x \in A_a$ define, $x^a = a \wedge x^{\sim}$. Then $(A_a, \wedge, \vee, {}^a)$ is a Pre A*-algebra with 1(here a is itself is the identity for $\wedge$ in A_a; that is 1 in A_a).

Proof: Let $x, y \in A_a$.Then $a \wedge x = x$ and $a \wedge y = y$.

Now $a \wedge (x \wedge y) = (a \wedge x) \wedge y = x \wedge y \Rightarrow x \wedge y \in A_a$

Also $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = x \vee y \Rightarrow x \vee y \in A_a$

Therefore A_a is closed under the operation $\wedge$ and $\vee$.

$a \wedge x^a = a \wedge (a \wedge x^{\sim}) = a \wedge x^{\sim} = x^a \Rightarrow x^a \in A_a$

Thus A_a is closed under ^{a}.

Now for any $x, y, z \in A_a$

(1) $x^{aa} = (a \wedge x^{\sim})^a = a \wedge (a \wedge x^{\sim})^{\sim} = a \wedge (a^{\sim} \vee x) = a \wedge x = x$

(2) $x \wedge x = (a \wedge x) \wedge (a \wedge x) = a \wedge x = x$

(3) $x \wedge y = (a \wedge x) \wedge (a \wedge y) = (a \wedge y) \wedge (a \wedge x) = y \wedge x$

(4) $(x \wedge y)^a = a \wedge (x \wedge y)^{\sim} = a \wedge (x^{\sim} \vee y^{\sim})$

$= (a \wedge x^{\sim}) \vee (a \wedge y^{\sim})$

$= x^a \vee y^b$

(5) $x \wedge (y \wedge z) = (a \wedge x) \wedge \{(a \wedge y) \wedge (a \wedge z)\}$

$= a \wedge \{x \wedge (y \wedge z)\}$

$= a \wedge \{(x \wedge y) \wedge z\}$ (since $x, y, z \in A$)

$= (x \wedge y) \wedge z$

(6) $x \wedge (y \vee z) = (a \wedge x) \wedge \{(a \wedge y) \vee (a \wedge z)\}$

$= \{(a \wedge x) \wedge (a \wedge y)\} \vee \{(a \wedge x) \wedge (a \wedge z)\}$

$= \{a \wedge (x \wedge y)\} \vee \{(a \wedge (x \wedge z)\}$

$= (x \wedge y) \vee (x \wedge z)$

(7) $x \wedge (x^a \vee y) = x \wedge \{(a \wedge x^{\sim}) \vee y\}$

$= \{x \wedge (a \wedge x^{\sim})\} \vee (x \wedge y)$

$= (x \wedge x^{\sim}) \vee (x \wedge y)$ (since $a \wedge x = x$)

$= x \wedge (x^{\sim} \vee y)$

$= x \wedge y$

Finally $x \in A_a$ implies that $a \wedge x = x = x \wedge a$. Thus $(A_a, \wedge, \vee, ^a)$ is a Pre A*-algebra with a as identity for $\wedge$.

2.10. Theorem: Let a, b be elements in a Pre A*-algebra A such that $a \leq b$.Then the following hold.

(1) $a \wedge b = a$

(2) The map $\alpha_{a,b} : A_b \to A_a$ defined by $\alpha_{a,b}$ (x) = $a \wedge x$ for all $x \in A_b$ is a homomorphism of Pre A*-algebras.

(3) $\alpha_{a,b}$ (B(A_b)) $\subseteq$ B(A_a)

(4)If $a \leq b \leq c$ then $\alpha_{a,b} \, o \, \alpha_{b,c} = \alpha_{a,c}$

(5) $\alpha_{a,a}$ is the identity map on A_a

Proof: Suppose that $a \leq b$

(1) We have $a \leq b \Rightarrow a \wedge b$ = a

(2) Let x, y $\in A_b$ .Then $\alpha_{a,b}$ $(x \wedge y) = a \wedge (x \wedge y)$

$= (a \wedge x) \wedge (a \wedge y)$

$= \alpha_{a,b}(x) \wedge \alpha_{a,b}(y)$

and $\alpha_{a,b}$ $(x \vee y) = a \wedge (x \vee y)$

$= (a \wedge x) \vee (a \wedge y)$

$= \alpha_{a,b}(x) \vee \alpha_{a,b}(y)$

Also $\alpha_{a,b}$ $(x^b) = a \wedge x^b$

$= a \wedge (b \wedge x^{\sim})$

$= (a \wedge b) \wedge x^{\sim}$

$= a \wedge x^{\sim}$

$= a \wedge (a^{\sim} \vee x^{\sim})$

$= a \wedge (a \wedge x)^{\sim}$

$= (a \wedge x)^a$

$= (\alpha_{a,b}(x))^a$

Therefore $\alpha_{a,b}$ is a homomorphism of Pre A*-algebras.

(3) Let $x \in$ B(A_b) .

Then $x \vee x^b = b$ (since b is identity in A_b) and therefore $b = x \vee (b \wedge x^{\sim})$

Now b = $b \wedge b$ = b $\wedge$ (x $\vee$ (b $\wedge$ $x^{\sim}$))

= (b $\wedge$ x) $\vee$ (b $\wedge$ $x^{\sim}$)

= b $\wedge$ (x $\vee$ $x^{\sim}$) -------------------(i)

Now $\alpha_{a,b}$ (x) $\vee$ [$\alpha_{a,b}$ (x)]^a = (a $\wedge$ x) $\vee$ $(a \wedge x)^a$

= (a $\wedge$ x) $\vee$ [a $\wedge$ $(a \wedge x)^{\sim}$]

= (a $\wedge$ x) $\vee$ [a $\wedge$ ($a^{\sim}$ $\vee$ $x^{\sim}$)]

= a $\wedge$ [x $\vee$ ($a^{\sim}$ $\vee$ $x^{\sim}$)]

= a $\wedge$ [$a^{\sim}$ $\vee$ (x $\vee$ $x^{\sim}$)]

= a $\wedge$ (x $\vee$ $x^{\sim}$)

= (a $\wedge$ b) $\wedge$ (x $\vee$ $x^{\sim}$)

= a $\wedge$ [b $\wedge$ (x $\vee$ $x^{\sim}$)]

= a $\wedge$ b (by (i))

= a, which is 1 in A_a

Therefore $\alpha_{a,b}$ (x) $\in$ B(A_a)

Thus $\alpha_{a,b}$ (B(A_b)) $\subseteq$ B(A_a)

(4)Let a $\leq$ b $\leq$ c

[$\alpha_{a,b}$ o $\alpha_{b,c}$](x) = $\alpha_{a,b}$ [$\alpha_{b,c}$(x)]

= $\alpha_{a,b}$ [b $\wedge$ x]

= a $\wedge$ b $\wedge$ x

= a $\wedge$ x

= $\alpha_{a,c}$ (x)

Therefore $\alpha_{a,b}$ o $\alpha_{b,c} = \alpha_{a,c}$

(5) $\alpha_{a,a}$ (x) = a $\wedge$ x = x for all x $\in$ A_a

Then $\alpha_{a,a}$ is identity map on A_a.

3. Conclusion

This manuscript illustrates the nature of the Pre-A*-algebra like a partially ordered set. With respect to binary operation $\wedge$, defined a relation $\leq$ on a Pre-A*-algebra and observed that such a Pre-A*-algebra as a partially ordered set with respect to the relation $\leq$ and derived corresponding results. It has been observed a necessary condition for a Pre-A*-algebra to become a lattice with respect to binary operation $\wedge$. For any a $\in$ A defined a set A_a = {x $\in$ A / a $\wedge$ x = x} and x^a = a $\wedge$ $x^{\sim}$, observed that (A_a, $\wedge$, $\vee$, ^{a}) is a Pre A*-algebra. Also by defining a mapping $\alpha_{a,b}$ from A_b to A_a by $\alpha_{a,b}$ (x) = a $\wedge$ x for all x $\in$ A_b, confirmed a homomorphism of Pre A*-algebras.

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