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# Singular Modified Riemann-Hilbert Problems for Nonlinear Elliptic Complex Equations of First Order

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**Abstract:** *In [1], the author first proposed a well-posedness of singular Riemann-Hilbert boundary value problem for generalized analytic functions in multiply con-nected domains, and the well posedness allows that the solutions of the modified problem possess some poles in N + 1-connected domain D. In [3], the author proposed another well-posedness of the Riemann-Hilbert boundary value problem with continuous solutions for nonlinear elliptic complex equations of first order, in particular the well-posedness includes the well-posedness of the singular case of 0 < K < N. Recently, the authors of this paper proposes three kinds of new well-posedness of singular Riemann-Hilbert boundary value problem for nonlinear elliptic complex equations of first order in multiply connected domains. We shall prove the existence of solutions for these boundary value problems.*

**Keywords:** *Singular modified Riemann-Hilbert problem, elliptic complex equations of first order, three kinds of well posedness with pole points, the existence of solutions.*

**AMS Mathematics Subject Classification:** *30J56, 35J46, 35J60.*

## 1. Formulation of Singular Modified Riemann-Hilbert Boundary Value Problems for Elliptic Complex Equations of First Order

First of all, we introduce the nonlinear elliptic equations of first order

$$\begin{cases} w_{\bar z} = F(z,w,w_z), F = Q_1 w_z + Q_2 \overline{w_z} + A_1 w + A_2 \bar w + A_3, \\ Q_j = Q_j(z,w,w_z), j = 1,2, A_j = A_j(z,w), j = 1,2,3, \end{cases} \quad (1.1)$$

In a bounded $N+1$ ($N \geq 1$)-connected domain $D$, which is the complex form of the real nonlinear elliptic system of first order equations

$\Phi j(x, y, u, v, u_x, u_y, v_x, v_y) = 0$, $j = 1, 2$

Under certain conditions(see Theorem 1.2, Chapter I, [4]). There is not harm in assuming that $D$ is an $N+1$ ($N \geq 1$)-connected circular domain in $|z| < 1$ bounded by the ($N+1$)- circles $\Gamma_j : |z - z_j| = r_j$, $j = 0, 1, ...., N$ and $\Gamma_0 = \Gamma_{N+1} : |z| = 1$, $z = 0 \in D$. In this article, the notations are as the same in References [3-13]. Suppose that the complex equation (1.1) satisfies the following conditions, namely

**Condition C.** 1) $Q_j(z, w, U)$ ($j = 1, 2$), $A_j(z, w)$($j = 1, 2, 3$) are measurable in $z \in D$ for all continuous functions $w(z)$ in $\overline{D}\{0\}$ and all measurable functions $U(z) \in L_{p_0}(\overline{D})$, and satisfy

$$L_p[A_j, \overline{D}] \leq k_0, \ j = 1, 2, \ L_p[A_3, \overline{D}] \leq k_1, \quad (1.2)$$

where $p$, $p_0$ ($2 < p_0 \leq p$), $k_0$, $k_1$ are non-negative constants.

2) The above functions are continuous in $w \in \mathbb{C}$ for almost every point $z \in D$, $U \in \mathbb{C}$, and $A_j = 0$ ($j$= 1, 2, 3) for $z \in \mathbb{C} \backslash D$.

3) The complex equation (1.1) satisfies the uniform ellipticity condition, i.e. for any $U_1$, $U_2 \in \mathbb{C}$,

the following inequality in almost every point $z \epsilon D$ holds:

$$|F(z, w, U_1) - F(z, w, U_2)| \leq q_0 |U_1 - U_2|, \quad (1.3)$$

In which $q_0$ (< 1) is a non-negative constant.

It is well known that a generalized analytic function in a domain $D$ is a continuous solution of the complex equation

$$w_{\bar{z}} = A(z)w + B(z)\overline{w},\ z \epsilon D, \quad (1.4)$$

Where $z = x + iy$, $w_{\bar{z}} = [w_x + i\ w_y]/2$, $A(z)$, $B(z)$ $\epsilon$ $L_p(\overline{D})$ ($p > 2$)*;* the conditions will be called Condition $C_0$. Obviously the complex equation (1.4) is a special case of (1.1).

Now we first formulate the new singular Riemann-Hilbert problem with the non-negative index for equation (1.1) as follows.

**Problem $B_1$.** The singular modified Riemann-Hilbert boundary value problem for (1.1) is to find a continuous solution $w(z)$ in $\overline{D}$ with the pole point of $n$ order at the point $z = 0$ ($\in D$) satisfying the boundary condition:

$$\mathrm{Re}\,[\overline{\lambda(z)}w(z)] = r(z) + h(z),\ z \epsilon \Gamma, \quad (1.5)$$

Where $\lambda(z)$, $r(z)$ satisfy the conditions

$C_\alpha[\lambda(z), \Gamma] \leq k_0$, $C_\alpha[r(z), \Gamma] \leq k_2$,in which
(1.6)

$\lambda(z) = a(z) + ib(z)$ on $\Gamma$ , $\alpha$ ($1/2 < \alpha < 1$) is a positive constant. The index $K$ of Problems $B_1$ is defined by:

$$K = K_0 + K_1 + \ldots\ldots + K_N = \sum_{j=0}^{N} \frac{1}{2\pi}\Delta_{\Gamma j} \arg \lambda(z) \geq 0, \quad (1.7)$$

The partial indexes $K_j = \Delta_{\Gamma j} \arg\ \lambda(z)/2\pi$ ($j = 0, 1, \ldots\ldots, N$) of $\lambda(z)$ are integers and

$$h(z)=\begin{cases} 0, z \epsilon \Gamma_0, \\ h_j, z \epsilon \Gamma_j, j = 1,\ldots\ldots, N, \end{cases} \quad (1.8)$$

$h_j$($j = 1, \ldots, N$) are unknown real constants to be determined appropriately. Moreover we assume that the solution $w(z)$ satisfies the following point conditions

$$\mathrm{Im}[\overline{\lambda(a_j)}w(a_j)] = b_j,\ j \epsilon J = \{1, \ldots\ldots, 2K+1\}, \quad (1.9)$$

in which $a_j \epsilon \Gamma_0$ ($j = 1, \ldots, 2K+1$) are distinct fixed points, and $b_j$($j \epsilon J$) are all real constants satisfying the conditions

$$|b_j| \leq k_3,\ j \epsilon J, \quad (1.10)$$

herein $k_3$ is a non-negative constant. Problem B with $A_3(z, w) = 0$ in $D$, $r(z) = 0$ on $\Gamma$ and $b_j$ ($j \epsilon J$) is called Problem $B_0$.

Next we shall introduce the other two kinds of well-posedness of new singular Riemann-Hilbert boundary value problem for the equation (1.1) as follows

**Problem $B_2$.** To find a continuous solution $w(z)$ of the equation (1.1) in $\overline{D}\backslash\{0\}$satis-fying the modified boundary conditions

$$\mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z) + h(z),\ z \epsilon \Gamma,$$

$$\mathrm{Im}[\overline{\lambda(a_j)}w(a_j)] = b_j,\ j \epsilon J = \{1, \ldots\ldots, 2K\},$$

$$w(0) = \infty,\ \ w(a) = 0,\ w(1) = 1, \quad (1.11)$$

where $a$ ($\epsilon$ $D$) is a point, and $\lambda(z)$, $r(z)$, $h(z)$ are the same as in (1.5)-(1.6), and $a_j$ ($\neq 1$) $\epsilon\Gamma_0$ ($j$=1,….., $2K$) are distinct fixed points, $b_j$($j \epsilon J$) are all real constants satisfying the conditions

$$|b_j| \leq k_3,\ j \epsilon J, \quad (1.12)$$

herein $k_3$ is a non-negative constant.

**Problem $B_3$.** To find a continuous solution $w(z)$ of the equation (1.1) in $\overline{D}\backslash\{0\}$ with the pole point of $n$ (> 0) order at $z = 0$ and the zero point of $m$ ($0 < m < n$) order at $z = a$ ($\epsilon D$, $a \neq 0$) satisfying the modified boundary conditions

$$\mathrm{Re}\,[\overline{\lambda(z)}w(z)] = r(z) + h(z),\ z \in \Gamma,$$

$$\mathrm{Im}[\overline{\lambda(a_j)}w(a_j)] = b_j, j \in J = \{1, \ldots, 2K+1\}, \tag{1.13}$$

in which $n$, $m$ ($< n$) are positive integers and $\lambda(z)$, $r(z)$, $h(z)$ are the same as in (1.5)-(1.6), and $a_j \in \Gamma_0$ ($j \in J = 1, \ldots, 2K + 1$) are distinct fixed points, $b_j$ ($j \in J$) are all real constants satisfying the condition

$$|b_j| \leq k_3,\ j \in J \tag{1.14}$$

with the constant $k_3$.

In order to prove the solvability of Problem $B_1$ for the complex equation (1.1), we need to give a representation theorem for Problem $B_1$.

**Theorem 1.1.** Suppose that the complex equation (1.1) satisfies Condition C, and $w(z)$ is a solution of Problem $B_1$ for (1.1). Then $w(z)$ is represented by

$$w(z) = [\Phi(\varsigma(z)) + \Psi(z)]e^{\Phi(z)}, \tag{1.15}$$

where $\varsigma(z)$ is a homeomorphism in $\overline{D}$, which quasiconformally maps D onto the $N + 1$-connected circular domain $G$ with boundary $L = \varsigma(\Gamma)$ in $\{|\varsigma| < 1\}$, such that $\varsigma(0) = 0$ and $\varsigma(1) = 1$, $\Phi(\varsigma)$ is an analytic function in G, $\Psi(z)$, $\Phi(z)$, $\varsigma(z)$ and its inverse function $z(\varsigma)$ satisfy the following estimates

$$C_\beta[\Psi, \overline{D}] \leq k_4,\ C_\beta[\Phi, \overline{D}] \leq k_4,\ C_\beta[\varsigma(z), \overline{D}] \leq k_4, \tag{1.16}$$

$$L_{p0}[|\Psi_{\bar z}| + |\Psi_z|, \overline{D}] \leq k_4;\ L_{p0}[|\Phi_{\bar z}| + |\Phi_z|, \overline{D}] \leq k_4, \tag{1.17}$$

$$C_\beta[z(\varsigma), \bar G] \leq k_4,\ L_{p0}[|\chi_{\bar z}| + |\chi_z|, \overline{D}] \leq k_5, \tag{1.18}$$

in which $\chi(z)$ is as stated in (1.21) below, $\beta = \min(\alpha, 1 - 2/p_0)$, $p_0$ ($2 < p_0 \leq p$), $k_j = k_j(q_0, p_0, \beta, k_0, k_1, D)$ ($j = 4, 5$) are non-negative constants dependent on $q_0$, $p_0$, $\beta$, $k_0$, $k_1$, $D$. Moreover, the function $\Phi[\varsigma(z)]$ satisfies the estimate

$$C_\delta[\varsigma^n\Phi[\varsigma(z)], \overline{D}] \leq M_1 = M_1(q_0, p_0, \beta, k, D) < \infty, \tag{1.19}$$

and T ($\leq \min(\alpha, 1 - 2/p_0)$), $k = k(k_0, k_1, k_2, k_3)$, and $M_1$ is a non-negative constant dependent on $q_0$, $p_0$, $\beta$, $k$, $D$. Here we mention that the pole of $n$ order at $z = 0$ of $w(z)$ is denoted the pole of $n$ order of the function $\Phi(\varsigma)$ at $\varsigma(0) = 0$.

**Proof.** We substitute the solution $w(z)$ of Problem $B_1$ into the coefficients of equation (1.1) and consider the following system

$$\Phi_{\bar z} = Q\Phi_z + A,\ A = \begin{cases} A_1 + A_2\overline{w}/w \text{ for } w(z) \neq 0, \\ 0 \text{ for } w(z) = 0 \text{ or } z \notin D, \end{cases}$$

$$\Psi_{\bar z} = Q\Psi_z + A_3\, e^{-\Phi(z)},\ Q = \begin{cases} Q_1 + Q_2\overline{w_z}/w_z \text{ for } w_z \neq 0, \\ 0 \text{ for } w_z = 0 \text{ or } z \notin D, \end{cases} \tag{1.20}$$

$$w_{\bar z} = QW_z,\ W(z) = \Phi[\varsigma(z)] \text{ in } D.$$

By using the continuity method and the principle of contracting mapping, we can find the solution

$$\Psi(z) = T_0 f = \frac{-1}{\pi}\iint_D \frac{f(\varsigma)}{\varsigma - z} d\sigma_\varsigma, \tag{1.20}$$

$$\Phi(z) = T_0 g,\ \varsigma(z) = \Psi[\chi(z)],\ \chi(z) = z + T_0 h$$

of (1.20), in which $f(z)$, $g(z)$, $h(z) \in L_{p0}(\overline{D})$, $2 < p_0 \leq p$, $\chi(z)$ is a homeomorphic solution of the third equation in (1.20), $\Psi(\chi)$ is a univalent analytic function, which con-formally maps $E = \chi(D)$ onto the domain $G$ (see[1,3], and $\Psi(\varsigma)$ is an analytic function in $G$ such that the function $\varsigma(z) = \Psi[\chi(z)]$ satisfies $\varsigma(0) = 0$, $\varsigma(1) = 1$. We can verify that $\Psi(z)$, $\Phi(z)$, $\varsigma(z)$ satisfy the estimates (1.16) and (1.17). It remains to prove that $z = z(\varsigma)$ satisfies the estimate in (1.18). In fact, we can find a homeomorphic solution of the last equation in (1.20) in the form $\chi(z) = z + T_0 h$ such that $[\chi(z)]_z$, $[\chi(z)]_{\bar z} \in L_{p0}(\overline{D})$ (see[1]). By the result on conformal mappings, applying the method of Theorem

3.2, Chapter V,[4], we can prove that (1.18) is true. It is easy to see that the function $\Phi|\varsigma(z)|$ satisfies the boundary conditions

$\mathrm{Re}[\overline{\lambda(z)}e^{\Phi(z)}\,\Phi(\varsigma(z))]=c(z)+h(z) - \mathrm{Re}[\overline{\lambda(z)}e^{\Phi(z)}\Psi(z)],\ z \in \Gamma$

On the basis of the estimates (1.16) and (1.18), and using the methods of Theorems 3.2–3.3, Chapter V, [3], we can prove that $\Psi[\varsigma(z)]$ satisfies the estimate (1.19).

## 2. Unique Solvability of Problems $B_1$, $B_2$, $B_3$ for Generalized Analytic Functions

In this section, we first prove the uniqueness and solvability of Problems $B_j$ (j = 1, 2, 3) for generalized analytic functions.

**Theorem 2.1.** Suppose that equation (1.4) satisfies Condition $C_0$. Then the solution of Problem $B_1$ are existence and unique

**Proof.** Problem $B_1$ for (1.4) can be rewritten as

$\mathrm{Re}[\overline{\lambda(z)}[1/z^n]W(z)] = r(z) + h(z)$ in D,

$\mathrm{Im}[\overline{\lambda(a_j)}\,[1/a_j^n]W(a_j)] = b_j',\ j \in J = \{1, \ldots, 2(K+n)+1\},$ (2.1)

Where $W(z) = w(z)/\Psi(z)$, $\Psi(z) = 1/z^n$, $b_j'$ $(j \in J)$ are real constants with the conditions

$|b_j'|$ - $k_3'$ $(<\infty)$ $(j \in J)$. It is easy to see $W(z)$ satisfies the complex equation

$W_{\bar z} = A(z)W + [B(z)\Psi(z) / \overline{\Psi(z)}]\overline{W},\ z \in D,$ (2.2)

The index of $\lambda(z)\ \overline{1/z^n}$ on $\Gamma$ is equal to $K+n$ (> 0), the boundary value problem (2.1),(2.2) is called Problem $B'_1$. According to the method a before, we can derive that Problem $B'_1$ has a unique continuous solution $W(z)$ in $\overline{D}$, and then Problem $B_1$ for (1.4) is uniquely solvable.

**Theorem 2.2.** Suppose that equation (1.4) satisfies Condition $C_0$. Then the solution of Problem $B_2$ are existence and unique.

**Proof.** Problem $B_2$ for (1.4) can be rewritten as

$\mathrm{Re}[\overline{\lambda(z)}\,[(z - a)/(1 - a)z]W(z)] = r(z) + h(z)$ in D,

$(1 - a)/(1 - a)]W(1) = 1,$ (2.3)

Where $W(z) = w(z)/\Psi(z)$, $\Psi(z) = (z - a)/(1 - a)z$. It is easy to see $W(z)$ satisfies the complex equation

$W_{\bar z} = A(z)W + [B(z)\Psi(z)/\overline{\Psi}]\overline{W},\ z \in D,$ (2.4)

the index of $\lambda(z)\overline{(z - a)}\ /\overline{(1 - a)z}$ on $\Gamma$ is equals to $K$, the boundary value problem (2.3),(2.4) and the second formula of (1.11) is called Problem $B_2'$, hence according to the result as in Theorem 3.3,Chapter V,[4], f we can derive that Problem $B_2'$ has a unique continuous solution $W(z)$ in $\overline{D}$, and then Problem $B_2$ for (1.4) is uniquely solvable.

**Theorem 2.3.** *Suppose that equation* (1.4) *satisfies Condition* $C_0$. *Then the solution of Problem* $B_3$ *is existence and unique.*

**Proof.** For problem $B_3$ for (1.4) can be rewritten as

$\mathrm{Re}[\overline{\lambda(z)}\,[(z - a)^m/z^n]W(z)] = r(z) + h(z)$ *in D,*

$\mathrm{Im}[\overline{\lambda(a_j)}\,[(a_j - a)^m/a^n_j]W(a_j)] = b_j',\ j \in J = \{1, \ldots, 2(n - m + K)+1\},$ (2.5)

Where $W(z) = w(z)/\Psi(z)$, $\Psi(z) = (z - a)^m/z^n$, $b_j'$ $(j \in J)$ are real constants with the conditions $|b'_j| \le k_3'$ $(< \infty)$ $(j \in J)$. It is easy to see $W(z)$ satisfies the complex equation

$W_{\bar z} = A(z)W + [B(z)\Psi(z)/\overline{\Psi}]\overline{W},\ z \in D,$ (2.6)

That the index of $\lambda(z)\overline{(z - a)}^m\ /\overline{z^n}$ on $\Gamma$ is equals to $K + n - m$(> 0), the boundary value problem (2.5),(2.6) is called Problem $B'_3$. Moreover we can derive that Problem $B_3'$has a unique continuous solution $W(z)$ in$\overline{D}$, and then Problem $B_3$ for (1.4) is uniquely solvable.

In the following section, by using Theorem 3.3, Chapter V,[4] ,we can prove the solvability of Problems $B_j$ (j = 1, 2, 3) for (1.1).

## 3. ESTIMATES OF SOLUTIONS AND SOLVABILITY OF PROBLEMS $B_j$ ($j$ = 1, 2, 3) FOR NONLINEAR ELLIPTIC COMPLEX EQUATIONS IN MULTIPLY CONNECTED DOMAINS

The singular modified Riemann-Hilbert problem(Problem $B_1$) can be transformed into the continuous modified Riemann-Hilbert problem(Problem $B_1'$) as follows.

**Problem B'$_1$.** The modified Riemann-Hilbert boundary value problem for (1.1) is to find a continuous solution w (z) in $\overline{D}$ satisfying the boundary condition:

$$\mathrm{Re}\,[\overline{\lambda(z)}\,/\varsigma^n w(z)] = r(z) + h(z),\ z \in \Gamma, \tag{3.1}$$

Where λ(z), r(z) satisfy the conditions

$$C_\alpha\,[\lambda(z), \Gamma\,] \le k_0,\ C_\alpha[r(z),\Gamma\,] \le k_2, \tag{3.2}$$

$\lambda(z) = a(z) + ib(z)$, $|\lambda(z)| = 1$ on $\Gamma$ , and $\alpha$ ($1/2 < \alpha < 1$) is a positive constant. The index K of Problems $B_1$ is defined as follows:

$$K + l = K_0 + K_1 + \ldots + K_N == \sum_{j=0}^{N} \frac{1}{2\pi}\Delta_{\Gamma_j} \arg \lambda(z) \ge 0, \tag{3.3}$$

The partial indexes $K_j = \Delta_{\Gamma_j} \arg \lambda(z)/2\pi$ of λ(z) are integers. And

$$h\,(z)= \begin{cases} 0, z \in \Gamma 0, \\ h_j, z \in \Gamma_j, j = 1,\ldots,N, \end{cases} \tag{3.4}$$

$h_j$(j = 1, …. , N) are unknown real constants to be determined appropriately. Moreover we assume that the solution w(z) satisfies the following point conditions

$$\mathrm{Im}[\overline{\lambda(a_j)}W\,(a_j)] = b_j,\ j \in J = \{1,\ \ldots,\ 2K + 2n + 1\}, \tag{3.5}$$

where $a_j \in \Gamma_0$ (j =1, …. , 2K + 2n + 1) are distinct fixed points; and $b_j$(j ϵ J) are all real constants satisfying the conditions

$$|b_j| \le k_3,\ j \in J, \tag{3.6}$$

herein $k_3$ is a non-negative constant.

**Theorem 3.1.** Suppose that the first order complex equation (1.1) satisfies Condition C. Then any solution w(z) of Problem $B_1$ for the complex equation (1.1) satisfies the estimates

$$C_\beta[\varsigma^n w(z), \overline{D}] \le M_1,$$

$$\hat{L}^1_{p_0}[w, \overline{D}] = L_{p0}\,[\,|[\varsigma^n w]_{\bar z}| + |[\varsigma^n w]_z|, \overline{D}] \le M_2, \tag{3.7}$$

in which $\beta = \min(\alpha, 1 - 2/p_0)$, $k = k(k_0, k_1, k_2, k_3)$, $M_j = M_j(q_0, p_0, \beta, k, D)$, (j = 1, 2) are positive constants.

**Proof.** Similarly to the proof of Theorem 1.1, the solution w(z) of Problem $B_1$ for (1.1) can be expressed the formula as in (1.15), hence the boundary value problem $B_1$ can be transformed into the boundary value problem (Problem $B_1$) for analytic functions

$$\mathrm{Re}[\overline{\Lambda(\varsigma)}\,\Phi(\varsigma)] = \hat{r}(\varsigma) + h(\varsigma),\ \varsigma \in L^* = \varsigma(\Gamma^*);$$

$$\mathrm{Im}[\overline{\Lambda\,(a_j')}\Phi(a_j')] = b_j',\ j \in J,\ a_j' \tag{3.8}$$

Where

$$h\,(\varsigma)= \begin{cases} 0, \varsigma \in L0, \\ h_j, \varsigma \in L_j, j = 1,\ldots,N, \end{cases}$$

And

$$\overline{\Lambda(\varsigma)} = \overline{\lambda[z(\varsigma)]}e^{\Phi[z(\varsigma)]},\ \hat{r}(\varsigma) = r[z(\varsigma)] - \mathrm{Re}\{\,\overline{\lambda[z(\varsigma)]}\Psi[z(\varsigma)]e^{\Phi[z(\varsigma)]}]\},$$

$$a_j' = \varsigma(a_j),\ \hat{b}_j - \mathrm{Im}[\overline{\lambda(a_j)}\Psi(a_j)],\ j \in J$$

By (1.5), (1.9), it can be seen that $\Lambda(\varsigma)$, $\hat{r}(\varsigma)$, $\hat{b}_j$ ($j \in J$) satisfy the conditions

$$C_{\alpha\beta}[\Lambda(\varsigma), L] \leq M_3,\ C_{\alpha\beta}[\hat{r}(\varsigma), L] \leq M_3,\ |\hat{b}_j| \leq M_3,\ j \in J, \tag{3.9}$$

Where $M_3 = M_3(q_0, p_0, \beta, k, D)$. If we can prove that the solution $\Phi(\varsigma)$ of Problem $\tilde{B}_1$ satisfies the estimate

$$C_{\alpha\beta}[\varsigma^n\Phi(\varsigma), \overline{G}] \leq M_4, \tag{3.10}$$

in which $G = \varsigma(D)$, $M_4 = M_4(q_0, p_0, \beta, k, D)$, then from the representation (3.3) of the solution $w(z)$ and the estimates about $\Phi(z)$, $\Psi(z)$, $\varsigma(z)$ and its inverse function $z(\varsigma)$, the estimates in (3.5) can be derived.

It remains to prove that (3.10) holds. For this, we first verify the boundedness of $\varsigma^n\Phi(\varsigma)$, i.e.

$$C[\varsigma^n\Phi(\varsigma), \overline{G}] \leq M_5 = M_5(q_0, p_0, \beta, k, D). \tag{3.11}$$

Suppose that (3.11) is not true. Then there exist sequences of functions $\{\Lambda_l(\varsigma)\}$, $\{\hat{r}_l(\varsigma)\}$, $\{\hat{b}_{jl}\}$ satisfying the same conditions as $\Lambda(\varsigma)$, $\hat{r}(\varsigma)$, $\hat{b}_j$ ,and $\Lambda_l(\varsigma)$, $\hat{r}_l(\varsigma)$, $\hat{b}_{jl}$ uniformly converge to $\Lambda_0(\varsigma)$, $\hat{r}_0(\varsigma)$, $\hat{b}_{j0}$ ($j \in J$) on $L$ respectively. For the solution $\Phi_l(\varsigma)$ of the boundary value problem (Problem $\widetilde{B_1}$) corresponding to $\Lambda_1(\varsigma)$, $\hat{r}_1(\varsigma)$, $\hat{r}_{j1}$ ($j \in J$) we have $I_l = C[\Phi_l(\varsigma), \bar{G}] \to \infty$ as $n \to \infty$. There is no harm in assuming that $I_l \geq 1$, $l = 1, 2, \ldots$. Obviously $\tilde{\Phi}_l(\varsigma) = \Phi_l(\varsigma)/I_l$ satisfies the boundary conditions

$$\mathrm{Re}[\overline{\Lambda_1(\varsigma)}\,\tilde{\Phi}_l(\varsigma)] = [\hat{r}_1(\varsigma) + h(\varsigma)]/I_l,\ \varsigma \in L^*,$$

$$\mathrm{Im}[\overline{\Lambda_1(a_j')}\,\tilde{\Phi}_l(a_l')] = \hat{b}_{jl}/ I_l,\ \ j \in J,$$

Applying the Schwarz formula, the Cauchy formula and the method of symmetric ex-tension (see Theorems 3.2-3.3, Chapter V, [3]), the estimate

$$C_{\alpha\beta}[\varsigma^n\tilde{\Phi}_l(\varsigma), \bar{G}] \leq M_6 \tag{3.12}$$

Can be obtained, where $M_6 = M_6(q_0, p_0, \beta, k, D)$. Thus we can select a subsequence of $\{\tilde{\Phi}_l(\varsigma)\}$, which uniformly converge to an analytic function $\tilde{\Phi}_0(\varsigma)$ in $G$, and $\tilde{\Phi}_0(\varsigma)$ satisfies the homogeneous boundary conditions

$$\mathrm{Re}[\overline{\Lambda_0(\varsigma)}\,\tilde{\Phi}_0(\varsigma)] = h(\varsigma),\ \varsigma \in L^*,$$

$$\mathrm{Im}[\overline{\Lambda_0(a_j')}\,\tilde{\Phi}_0(a_j')] = 0,\ \ j \in J,$$

On the basis of the uniqueness theorem, we conclude that $\tilde{\Phi}_0(\varsigma) = 0$, $\varsigma \in \bar{G}$. However, $C[\varsigma^n\tilde{\Phi}_l(\varsigma), \bar{G}]=1$ from $C[\varsigma^n\tilde{\Phi}_1(\varsigma), \bar{G}]=1$ ,it follows that there exists a point $\varsigma_* \in \bar{G}$; such that $C[\varsigma_*^n\tilde{\Phi}_0(\varsigma_*)|=1$, This contradiction proves that (3.11) holds. Afterwards using the method which leads from $C[\varsigma^n\tilde{\Phi}_l(\varsigma), \bar{G}]=1$ to (3.12), the estimate (3.7) can be derived.

For verifying the existence of solutions of Problem $B_1$ for the complex equation (1.1), we need to add the following condition. For any continuous functions $w_1(z)$, $w_2(z)$ in $\overline{D\backslash\{0\}}$ and $[\varsigma(z)]^nU(z) \in L_{p0}(\bar{D})$, there is

$$F(z, w_1, U) - F(z, w_2, U) = \tilde{Q}(z, w_1, w_2, U)U + \tilde{A}(z, w_1, w_2, U)(w_1 - w_2), \tag{3.13}$$

where $|\tilde{Q}(z, w_1, w_2, U) \leq q_0(< 1)$, $L_p[\tilde{A}(z, w_1, w_2, U), \bar{D}] \leq k_0$.When (1.1) is linear, (3.1) obviously holds. Moreover we first prove the existence of solutions of Problem $B_1$ for equation (1.1) with $F(z, w, w_z) = 0$ in $D_{1/m} = \{|z| < 1/m\}\ U\ \{|z - a| < 1/m\}$, i.e.

$$w_{\bar{z}} = F_{1/m}(z, w, w_z),\ F_{1/m}(z, w, w_z) = \begin{cases} F(z,w,w_z), z \in D_m = D\backslash D_{1/m} \\ 0, z \in D_{1/m} \end{cases} \tag{3.14}$$

By the Leray-Schauder theorem, where $m$ is a sufficiently large positive integer.

**Theorem 3.2.** *Suppose that equation* (1.1) *satisfies Condition* C *and* (3.13). *Then the singular Riemann-Hilbert problem* (*Problem* $B_1$) *for* (3.14) *has a solution.*

**Proof.** In order to find a solution $w(z)$ of Problem $B_1$ for equation (3.14), we consider the equation (3.14) with the parameter $t \in [0, 1]$

$w_{\bar{z}} = tF(z, w, w_z)$, $F(z, w, w_z) = Q_1 w_z + Q_2 \overline{w_{\bar{z}}} + A_1 w + A_2 \overline{w} + A_3$ in D, (3.15)

and introduce a bounded open set $B_M$ of Banach space $B = C_\beta(D_m) \cap L^1_{p0}(D_m)$, whose elements are functions $w(z)$ satisfying the condition

$$w(z) \in C_\beta(D_m) \cap L^1_{p0}(D_m) : C_\beta[w, D_m] + L^1_{p0}[w, D_m]$$

$$= C_\beta[w(z), D_m] + L_{p0}[\,|w_{\bar{z}}| + |w_z|, D_m] < M_7, \tag{3.16}$$

where $M_7 = 1 + M_1 + M_2$, $M_1$, $M_2$, $\beta$ are constants as similar to (3.7). We choose an arbitrary function $W(z) \in \overline{B_M}$ and substitute it in the position of $w$ in $F(z, w, w_z)$, Applying the method in the proof of Theorem 1.1.2, [12], a solution $w(z) = \Phi(z) + \Psi(z) = W(z) + T(tF)$ of Problem $B_1$ for the complex equation

$$w_{\bar{z}} = tF(z, W, W_z) \tag{3.17}$$

Can be found. Noting that $tF[z, W(z), W_z] \in L_{p0}(\overline{D})$, the above solution of Problem $B_1$ for (3.17) is unique. Denoting by $w(z) = \tilde{T}[W, t]$ $(0 \le t \le 1)$ the mapping from $W(z)$ to $w(z)$, from Theorem 3.2, we know that if $w(z)$ is a solution of Problem B for the equation

$$w_{\bar{z}} = tF(z, w, w_z) \text{ in } D, \tag{3.18}$$

then the function $w(z)$ satisfies the estimate

$$C_\beta[w, D_m)] < M_7 \tag{3.19}$$

Set $B_0 = B_M \times [0, 1]$. In the following we verify the three conditions of the Leray-Schauder theorem:

(1) For every $t \in [0, 1]$, $\tilde{T}[W, t]$ continuously maps the Banach space $B$ into itself, and is completely continuous in $\overline{B_M}$. In fact, we arbitrarily select a sequence $W_n(z)$ in $\overline{B_M}$, $n = 0, 1, 2, \ldots$, such that $C_\beta[W_n - W_0, D_m] \to 0$ as $n \to \infty$. By Condition C, we see that $L_{p0}[F(z, W_n, W_{nz}) - F(z, W_0, W_{0z})), \overline{D}] \to 0$ as $n \to \infty$. Moreover, from $w_n = \tilde{T}[W_n, t]$, $w_0 = \tilde{T}[W_0, t]$, it is easy to see that $w_n - w_0$ is a solution of Problem B for the following complex equation

$$(w_n - w_0)_{\bar{z}} = t[F(z, W_n, W_{nz}), - F(z, W_0, W_{0z})] \text{ in } D, \tag{3.20}$$

and then we can obtain the estimate

$$C_\beta[w_n - w_m, D_m)] \le 2k_0 C_\beta[W_n(z) - W_0(z), D_m]. \tag{3.21}$$

Hence $C_\beta[w_n - w_0, D_m] \to 0$ as $n \to \infty$. In addition for $W_n(z) \in \overline{B_M}$, $n = 1, 2, \ldots$, we have $w_n = \tilde{T}[W_n, t]$, $w_m = \tilde{T}[W_m, t]$, $W_n, W_m \in \overline{B_M}$, and then

$$(w_n - w_m)_{\bar{z}} = t[F(z, W_n, W_{nz}) - F(z, W_m, W_{mz})] \text{ in } D, \tag{3.22}$$

Where $L_{p0}[F(z, W_n, W_{nz}) - F(z, W_m, W_{mz}), \overline{D}] \le 2k_0 M_7$. Hence similarly to the proof of Theorem 3.1, we can obtain the estimate

$$C_\beta[w_n - w_m, D_m] \le M_7 M_8,$$

Where $M_8 = M_8(q_0, p_0, \beta, k, D)$. Thus there exists a function $w_0(z) \in \overline{B_M}$, from $\{w_n(z)\}$ we can choose a subsequence $\{w_{nk}(z)\}$ such that $C_\beta[w_{nk} - w_0, D_m] \to 0$ as $k \to \infty$. This shows that $w = \tilde{T}[W, t]$ is completely continuous in $\overline{B_M}$. Similarly we can prove that for $W(z) \in \overline{B_M}$, $\tilde{T}[W, t)$ is uniformly continuous with respect to $t \in [0, 1]$.

(2) For t = 0, it is evident that $w = \tilde{T}[W, 0] = \Phi(z) \in B_M$.

(3) From the estimate (3.7), we see that $w = \tilde{T}[W, t]$ $(0 \le t \le 1)$ does not have a solution $w(z)$ on the boundary $\partial B_M = \overline{B_M} \setminus B_M$.

Hence by the Leray-Schauder theorem, we know that there exists a function $w(z) \in \overline{B_M}$, such that $w(z) = \tilde{T}[w(z), t]$, and the function $w(z) \in C_\beta(D_m)$ is just a solution of Problem $B$ for the complex equation (3.14).

**Theorem 3.3.** Suppose that equation (1.1) satisfies Condition C and (3.13). Then Problem $B_1$ for (1.1) have a solution.

**Proof.** According to Theorem 3.2, we have proved that Problem $B_1$ for (3.14) have a solution $w_{1/m}(z)$, let $m \to \infty$, we can derive that $w_0(z)$ is the solution Problem $B_1$ for (1.1).

**Theorem 3.4.** Suppose that equation (1.1) satisfies Condition C and (3.13). Then Problem $B_j$ (j=2, 3) for (1.1) have a unique solution.

**Proof.** We first verify the unique solvability of Problem $B_3$ for (1.1). As stated in the proof of Theorem 2.3, the boundary conditions (1.13) can be reduced to the following boundary conditions

$\mathrm{Re}[\overline{\lambda(z)}\,[(\varsigma - \varsigma(a))^m/\varsigma^n]W(z)] = r(z) + h(z)$ in D,

$\mathrm{Im}[\overline{\lambda(a_j)}[\varsigma(a_j) - \varsigma(a))^m/(\varsigma(a_j))^n]W(a_j)] = b_j'$, $j \in J$,

Where $W(z) = w(z)/\Psi(z)$, $\Psi(z) = (\varsigma - \varsigma a))^m/\varsigma^n$, $b'_j$ $(j \in J)$ are real constants. It is easy to see $W(z)$ satisfies the complex equation

$w_{\bar{z}}=Q_1W_z + Q_2\bar{w}_{\bar{z}}-[Q_1\Psi'(z)-A]W - [Q_2\overline{\Psi'(z)} - B(z)\Psi(z)/\bar{\Psi}]\bar{W} + A_3\Psi(z)$, $z \in D$,

which index of $\lambda(z)\overline{(\varsigma(z) - \varsigma(a))^m}/\varsigma^n$ on $\Gamma$ equals to $K + n - m (> 0)$, by Theorem 3.3, the solvability of the boundary value problem (1.13) for (1.1) is verified.

Similarly we can prove the solvability of Problem $B_2$ for (1.1). From the solvability of Problem $B_2$ for (1.1), we can derive the existence of the homeomorphic solution for the nonlinear complex equation (1.1) with $A(z, w) = B(z, w) = C(z, w) = 0$ in $D$ from the domain $D$ mapping to the $N$+1 - connected rectilinear slit domain $G$, the so-called $N$+1 - connected rectilinear slit domain means a domain whose boundary consists of $N$ +1 rectilinear slits $L_j$ $(j = 0, 1, \ldots, N)$ with the oblique angles $\theta_j$ $(j = 0, 1, \ldots, N)$ respectively, where we must choose $\lambda(z) = e^{-i(\arg\,\theta j+\pi/2)}$, $\theta_j$ $(j = 0, 1, \ldots, N)$ are real constants, in this case, the index $K = 0$.

Finally we give the conclusion in this paper, namely the singular Riemann-Hilbert problem with the nonnegative index for elliptic complex equations of first can be trans-formed into the non-singular Riemann-Hilbert problem with the nonnegative index for the corresponding complex equations of first order, due to we can handle the non-singular boundary value problem, then the corresponding results of non-singular boundary value problem can be derived.

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