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A New Result on Generalized Summability Factors Via Convex Sequences

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Abstract: *In this paper, a known theorem dealing with* $|C,\alpha,\beta,\delta|_k$ *-summability factors has been generalized for* $|C,\alpha,\beta,\gamma,\delta|_k$ *-summability factors. Our theorem is based on some known results.*

1. Introduction

Let Σa_n be a given infinite series with partial sums (s_n). We denote by $u_n^{\alpha,\beta}$ and $t_n^{\alpha,\beta}$ the n-th Cesaro means of oprder (α,β), with $\alpha+\beta>-1$ of the sequence (s_n) and (na_n) respect ively (Browein [4]).

$$u_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=0}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}s_v \tag{1.1}$$

$$t_n^{\alpha,\beta}=\frac{1}{A_n^{\alpha+\beta}}\sum_{v=0}^{n}A_{n-v}^{\alpha-1}A_v^{\beta}va_v \tag{1.2}$$

where $A_n^{\alpha+\beta}=O(n^{\alpha+\beta}), A_0^{\alpha+\beta}=1$, and $A_n^{\alpha+\beta}=0$ for $n<0$.

The series Σa_n is said to be summable $|C,\alpha,\beta|_k, k\ge 1$ if (Das [6])

$$\sum_{n=1}^{\infty}n^{k-1}|u_n^{\alpha,\beta}-u_{n-1}^{\alpha,\beta}|^k<\infty \tag{1.3}$$

Since (Das [6]) $t_n^{\alpha,\beta}=n(u_n^{\alpha,\beta}-u_{n-1}^{\alpha,\beta})$ then

$$\sum_{n=1}^{\infty}n^{k-1}|u_n^{\alpha,\beta}-u_{n-1}^{\alpha,\beta}|^k=\sum_{-n=1}^{\infty}\frac{1}{n}|t_n^{\alpha,\beta}|^k<\infty \tag{1.4}$$

The series Σa_n is summable $|C,\alpha,\beta,\delta|_k, k\ge 1$ and $\delta\ge 0$ if (Bor [1])

$$\sum_{n=1}^{\infty}n^{\delta k+k-1}|u_n^{\alpha,\beta}-u_{n-1}^{\alpha,\beta}|^k=\sum_{-n=1}^{\infty}n^{\delta k-1}|t_n^{\alpha,\beta}|^k<\infty \tag{1.5}$$

And Σa_n is summable $|C,\alpha,\beta,\gamma,\delta|_k, k\ge 1, \delta\ge 0$ and $\gamma\ge 1$ if

$$\sum_{n=1}^{\infty}n^{\gamma(\delta k+k-1)}|u_n^{\alpha,\beta}-u_{n-1}^{\alpha,\beta}|^k=\sum_{-n=1}^{\infty}n^{\gamma(\delta k-1)}|t_n^{\alpha,\beta}|^k<\infty \tag{1.6}$$

If we take $\gamma=1$ then $|C,\alpha,\beta,\gamma,\delta|_k$ -summability reduces to $|C,\alpha,\beta,\delta|_k$ -summability. If we take $\gamma=1,\delta=0,\beta=0$ then $|C,\alpha,\beta,\gamma,\delta|_k$ -summability reduces to $|C,\alpha|_k$ -summability.

A sequence (λ_n) is said to be convex sequence if $\Delta^2\lambda_n>0$ where $\Delta^2\lambda_n=\Delta\lambda_n-\Delta\lambda_{n+1}$ and $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$.

2. KNOWN THEOREM

Bor [2] has proved the following theorem

Theorem 2.1 If (λ_n) is a convex sequence such that $\Sigma n^{-1}\lambda_n$ is convergent and the sequence $(W_n^{\alpha,\beta})$ defined by

$$W_n^{\alpha,\beta} = | t_n^{\alpha,\beta} |, \ \alpha = 1, \ \beta > -1 \tag{2.1}$$

$$W_n^{\alpha,\beta} = \max_{1\le v\le n} | t_n^{\alpha,\beta} |, \ 0 < \alpha < 1, \ \beta > -1 \tag{2.2}$$

Satisfying the condition

$$(n^{\delta} W_n^{\alpha,\beta})^k = O\{(\log n)^{p+k-1}\} \ (C,1) \tag{2.3}$$

Then the series $\Sigma(\log(n+1))^{-(p+k-1)} a_n\lambda_n$ is summable $|C,\alpha,\beta,\delta|_k$ for $0<\alpha\le 1$, $\beta>-1$, $k\ge 1$, $\delta\ge 0$, $p\ge 0$ and $\alpha+\beta-\delta>0$.

3. THE MAIN RESULT

Generalizing theorem 2.1 we have proved the following theorem.

Theorem 3.1 If (λ_n) is convex sequence such that $\Sigma n^{-1}\lambda_n$ is convergent and sequence $(W_n^{\alpha,\beta})$ defined by (2.1) and (2.2) satisfying the condition

$$(n^{\gamma(\delta k-1)+1} (W_n^{\alpha,\beta})^k) = O\{(\log n)^{p+k-1}\} \ (C,1)$$

then the series $\Sigma \log(n+1)^{-(p+k-1)} a_n\lambda_n$ is summable $|C,\alpha,\beta,\gamma,\delta|_k$ for $0<\alpha\le 1$, $\beta>-1$, $k\ge 1$, $\delta\ge 0$, $\gamma\ge 1$, $p\ge 0$ and $\alpha+\beta-\gamma(\delta-1)>0$.

4. LEMMAS

We need the following lemmas for the the proof of our theorem.

Lemma 4.1 (Chow [5]) If (λ_n) is a convex sequence such that the series $\Sigma n^{-1}\lambda_n$ is convergent, then (λ_n) is non-negative and non-increasing,

$$n\Delta\lambda_n = O(1) \ \text{as} \ n\to\infty$$

and

$$\lambda_n \log n = O(1) \ \text{as} \ n\to\infty$$

Lemma 4.2 (Bor [3]) If $0<\alpha\le 1, \beta>-1$ and $1\le v\le n$ then

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p\right| \le \max_{1\le m\le v} \left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p\right|$$

Lemma 4.3 (Prasad [8]) If $((\log(n+1))^{p+k-1} X_n)$ satisfies the same condition as (λ_n) in lemma 4.1 then

$$n(\log(n+1))^{p+k-1}\Delta X_n = O(1) \ \text{as} \ n\to\infty$$

and

$$\sum_{n=1}^{\infty} n(\log(n+1))^{p+k-1}\Delta^2 X_n = O(1) \ \text{as} \ m\to\infty$$

Lemma 4.4 (Lal [7]) If (λ_n) is a convex sequence such that the $\Sigma n^{-1}\lambda_n$ is convergent then for $p\ge 0$ and $k\ge 1$

$$\sum_{n=1}^{\infty} \frac{\Delta(\lambda_n)^k}{(\log(n+1))^{p(k+1)+(k-1)^2}} = O(1) \ \text{as} \ m\to\infty$$

5. PROOF OF THE THEOREM

We write

$$X_n = \frac{\lambda_n}{(\log(n+1)^{p+k-1}} = (\log(n+1))^{-(p+k-1)}$$

Let $(T_n^{\alpha,\beta})$ be the n-th (C,α,β) mean of the sequence $(\text{na}_n X_n)$ then

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-1}^{\alpha-1} A_v^{\beta} v a_v X_v$$

By Abel's transformation and using lemm 4.2, we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta X_v \sum_{i=1}^{v} A_{n-u}^{\alpha-1} A_i^{\beta} i a_i + \frac{X_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v$$

$$|T_n^{\alpha,\beta}| \le \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta X_v \left| \sum_{i=1}^{v} A_{n-i}^{\alpha-1} A_i^{\beta} i a_i \right| + \frac{X_n}{A_n^{\alpha+\beta}} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \right|$$

$$\le \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} W_v^{\alpha,\beta} \Delta X_v + X_n W_n^{\alpha,\beta}$$

$$= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta} \quad \text{(say)}$$

Since

$$|T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}| \le 2^k (|T_{n,1}^{\alpha,\beta}|^k + |T_{n,2}^{\alpha,\beta}|^k)$$

In order to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\gamma(\delta k-1)} |T_{n,r}^{\alpha,\beta}|^k < \infty \text{ for } r = 1, 2.$$

whenever $k > 1$, we can apply Hölder's inequality with k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$ we get that

$$\sum_{n=2}^{m+1} n^{\gamma(\delta k-1)} \left| T_{n,1}^{\alpha,\beta} \right|^k < \sum_{n=2}^{m+1} n^{\gamma(\delta k-1)} \left| \frac{1}{A_n^{\alpha+\beta}} \sum_{v=2}^{n-1} A_v^{\alpha} A_v^{\beta} W_v^{\alpha+\beta} \Delta X_v \right|^k$$

$$= \mathrm{O}(1) \sum_{n=2}^{m+1} \frac{n^{\gamma(\delta k-1)}}{n^{(\alpha+\beta)k}} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} v^{\beta k} \Delta X_v (W_v^{\alpha+\beta})^k \right\} \left\{ \sum_{v=1}^{n-1} \Delta X_v \right\}^{k-1}$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta X_v (W_v^{\alpha+\beta})^k \sum_{n=v+1}^{m+1} \frac{1}{n^{\gamma+(\alpha+\beta-\delta\gamma)k}}$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta X_v (W_v^{\alpha+\beta})^k \int_v^{\infty} \frac{dx}{x^{\gamma+(\alpha+\beta-\delta\gamma)k}}$$

$$= \mathrm{O}(1) \sum_{v=1}^{m} \Delta X_v v^{\gamma(\delta k-1)+1} (W_v^{\alpha,\beta})^k$$

$$= \mathrm{O}(1) \sum_{v=1}^{m+1} \Delta(\Delta X_v) \sum_{p=1}^{v} (p^{\gamma(\delta k-1)+1} (W_p^{\alpha,\beta})^k + \mathrm{O}(1) \Delta X_m \sum_{v=1}^{m} v^{\gamma(\delta k-1)+1} (W_v^{\alpha,\beta})^k$$

$$= \mathrm{O}(1) \sum_{v=1}^{m-1} \mathrm{v}(\log(\mathrm{v}+1))^{p+k-1} \Delta^2 X_v + \mathrm{O}(m(\log(m+1))^{p+k-1} \Delta \mathrm{X}_m)$$

$$= \mathrm{O}(1) \text{ as } m \to \infty$$

By the application of lemm 4.3 similarly, we have that

$$\sum_{n=1}^{m} n^{\gamma(\delta k-1)} \left| X_n W_n^{\alpha+\beta} \right|^k = \mathrm{O}(1) \sum_{n=1}^{m} \frac{X_n^k}{n} n^{\gamma(\delta k-1)+1} (W_n^{\alpha,\beta})^k$$

$$= \mathrm{O}(1) \sum_{n=1}^{m-1} \Delta(n^{-1} X_n^k) \sum_{v=1}^{n} v^{\gamma(\delta k-1)+1} (W_n^{\alpha,\beta})^k + \mathrm{O}(1) \frac{X_m^k}{m} \sum_{v=1}^{m} v^{\gamma(\delta k-1)+1} (W_n^{\alpha,\beta})^k$$

$$= \mathrm{O}(1) \sum_{n=1}^{m-1} n(\log(n+1))^{p+k-1} \Delta(n^{-1} X_n^k) + \mathrm{O}(X_m^k (\log(m+1))^{p+k-1})$$

$$= \mathrm{O}(1) \sum_{n=1}^{m-1} n^{-1} X_n^k (\log(n+1))^{p+k-1} + \mathrm{O}(1) \sum_{n=1}^{m-1} n^{-1} X_n^k (\log(n+1))^{p+k-1} \Delta X_n^k$$

$$+O(1)((\log(m+1))^{p+k-1}X_m^k)$$

$$= O(1)\sum_{n=1}^{m-1}\frac{(\lambda_n \log(n+1))^k}{(n+1)(\log(n+1))^{1+p(k-1)+k(k-1)}} + O(1)\sum_{n=1}^{m-1}\frac{\Delta\lambda_n^k}{(\log(n+1))^{p(k-1)+(k-1)^2}}$$

$$+O(1)\left(\frac{(\lambda_m \log(m+1))^k}{(\log(m+1))^{p(k-1)+k(k-1)+1}}\right)$$

$$= O(1) \text{ as } m \to \infty$$

By the application of lemm 4.4.

This completes the proof of the theorem.

6. Conclusion

Above theorem gives the more general results in comparision of the theorem of H.Bor and will have an important place in the existing literature.

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References

[1] Bor, H.; On the generalized absolute Cesaro summability, Pac. J. Appl. Math. (2009).
[2] Bor, H.; A new factor theorem for absolute Cesaro summability, Bul. Of Math. Analysis and Appl. Vol. 3 (2011).
[3] Bor, H.; On a new application of quasi power increasing sequences, Proc. Est. Acad. Sci. 57, (2008).
[4] Borwein, D.; Theorems on some methods of summability Quart. J. Math. Oxford, Ser. 9 (1958).
[5] Chow, H.C.; On the summability factor of Fourier series, J. London Math. Soc. 16 (1941).
[6] Das, G.; A Tauberian theorem for absolute summability, Proc. Camb. Phil. Soc. 67 (1970).
[7] Lal, S.N.; On the absolute summability factors in infinite series, Bull. Acad. Polon. Sci. Math. Astronom. Phys., 17 (1969).
[8] Prasad, B.N., Bhatt, S.N.; The summability factor of a Fourier series, Duke Math J., 24 (1957).

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