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Geodetic Computations on Triaxial Ellipsoid

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Abstract: *Rotational ellipsoid generally used in geodetic computations. Triaxial ellipsoid surface although a more general so far has not been used in geodetic applications and , the reason for this is not provided as a practical benefit in the calculations. We think this traditional thoughts ought to be revised again. Today increasing GPS and satellite measurement precision will allow us to determine more realistic earth ellipsoid. Geodetic research has traditionally been motivated by the need to continually improve approximations of physical reality. Several studies have shown that the Earth, other planets, natural satellites,asteroids and comets can be modeled as triaxial ellipsoids. In this paper we study on the computational differences in results,fitting ellipsoid, use of biaxial ellipsoid instead of triaxial elipsoid, and transformation Cartesian (Geocentric ,Rectangular) coordinates to Geodetic coodinates or vice versa on Triaxial ellipsoid.*

Keywords: *Reference Surface, Triaxial ellipsoid, Coordinate transformation,Cartesian, Geodetic, Ellipsoidal coordinates*

1. INTRODUCTION

Although Triaxial ellipsoid equation is quite simple and smooth but geodetic computations are quite difficult on the Triaxial ellipsoid. The main reason for this difficulty is the lack of symmetry. Triaxial ellipsoid generally not used in geodetic applications. Rotational ellipsoid (ellipsoid revolution ,biaxial ellipsoid , spheroid) is frequently used in geodetic applications . Triaxial ellipsoid is although a more general surface so far but has not been used in geodetic applications. The reason for this is not provided as a practical benefit in the calculations. We think this traditional thoughts ought to be revised again.

Geodetic research has traditionally been motivated by the need to approximate closer and closer the physical reality. Several investigations have shown that the earth is approximated better by a triaxial ellipsoid rather than a biaxial one.

First, reference surface, the basic definitions ellipse, rotating ellipsoid and triaxial ellipsoid will introduce some mathematical equations to explaining the concepts. To show how transformation occur you will find two numerical examples related on this subject cartesian - geodetic coordinates transformation on Triaxial ellipsoid. The main aim of this study was to investigate of the computational differences in results, use of biaxial ellipsoid instead of triaxial elipsoid.

1.1. Which Reference Surface?

Geodesic studies need to accept a model for the earth. This model can not be the intersection of the atmosphere with the earth. The oceans shows a partial regularity, but we can not talk of such a scheme on land. The reference surface for the earth to be selected according to the actual shape of the earth, measurement and calculation must be easy to connect with it. The most suitable physical model for the earth is geoid. But geoid is not a suitable surface for geodetic computations. Because geoid has undulating shape, has no symmetry and can not be described as math. Therefore reference surface is selected as appropriate geometric surfaces to Earth. Selection of the reference surface depend on the size of the region study and desired accuracy.

The reference ellipsoids are used as a preferred surface on which geodetic network computations are performed and point coordinates such as latitude, longitude, and elevation are defined.The reference surfaces can be selected for the Earth from simple to complex: Plane(tangent),Sphere,Ellipsoid (biaxial) and Triaxial ellipsoid

Ease of Calculation

$$\longrightarrow$$

Earth>>Geoid>>Triaxial Ellipsoid>>Biaxial Ellipsoid >>Sphere>>Tangent Plane

$$\longleftarrow$$

Reality-Accuracy

If we think of asking a question about the above ranking accuracy computations will be difficult. If desire to simplify on the computational, the accuracy will be reduced. Considering today's computing facilities can not talk about the difficulty computing.

In addition, triaxial ellipsoid surface is most suitable reference surface for Earth and triaxial ellipsoid is more general surface than the rotational ellipsoid. Triaxial ellipsoid formulas are quite useful, because obtaining the rotational ellipsoid formula from Triaxial ellipsoid formula is easy. For this, equatorial semi-axis are accepted equal to each other (a_x=a_y =a) which is sufficient on triaxial ellipsoid formula. Similarly to obtain sphere formula from rotational ellipsoid formula it is sufficient to take as (a = b= R). And to obtain plain formula from sphere Formula it is sufficient to take as (R= ∞) is sufficient **[1]**.

According to common thought of today, biaxial ellipsoid has two different semi-axis which are most suitable for geometric models to the world in geodetic applications. Geodetic applications are generally rotational elipsoid and are use instead of triaxial ellipsoid which will be enough for today's needs.

Geodetic applications in the near future will be used for triaxial ellipsoid or more advanced surfaces are expected. Today, even if triaxial ellipsoid is not use for the world, we think it will be use for other celestial bodies and other aplications such as; image processing, face recognition, computer games etc.

The celestial bodies whose shapes are different from a sphere or ellipsoid it is possible to use the well known triaxial ellipsoid as a reference surface which is mathematically calculated and shows the elongated figure of a body. Many small bodies such as satellites, asteroids or nuclei of comets have more complex figures than a triaxial ellipsoid can show. In accordance of investigations of Academician Lyapunov A.N.”The celestial bodies have most stable shapes close to triaxial ellipsoid”. Traditionally most non-spherical bodies are approximated by a triaxial ellipsoid which is mathematically calculated and shows the non-spherical figure of the body **[8]**.

Despite that the commonly used Earth reference systems, like WGS-84, are based on rotational ellipsoids, there have also been over the course of the years permanent scientific investigations undertaken into different aspects of the triaxial ellipsoid. Examples include Lamé surfaces as generalization of triaxial ellipsoids, the triaxiality of the Earth and other celestial bodies **[2]**, determination of orbit view-periods from a station on a triaxial ellipsoid , gravity coupling in a triaxial Earth**[10]**. According to **[7]**, see also therein Table 1, the difference between the Earth’s equatorial semimajor and semiminor axis is ≈70 m. It is unnecessary to say that nowadays geodetic measurement techniques provide ellipsoidal height values with accuracies much better than this value. The problem of coordinate transformation is one of the most important issues in the geodetic calculations on the triaxial ellipsoid. The problem of the transformation from Cartesian to geodetic coordinates on triaxial ellipsoid were discussed by several authors ; The problem has been recently discussed by **[3]**, **[6]** , **[8]**

2. BASİC DEFİNİTİON

Ellips, Rotational Ellipsoid, Triaxial Ellipsoid:

An ellipse a curved line forming a closed loop, where the sum of the distances from two points (foci) to every point on the line is constant, Show the Fig.1.

Rotational ellipsoid used in geodesy is obtained, an ellips by rotating 180° around the vertical Z-axis (minor semi-axis) in Fig.2.

The standart equation of an elips and rotational ellipsoid centered at the origin of a cartesian coordinate system and aligned with the axes are given above.

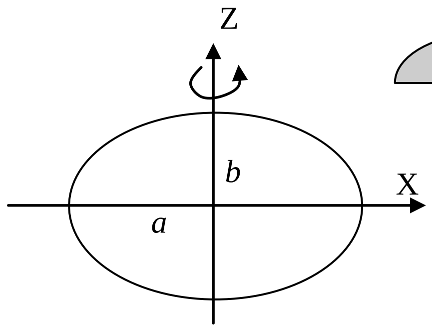

Figure-1 Ellips

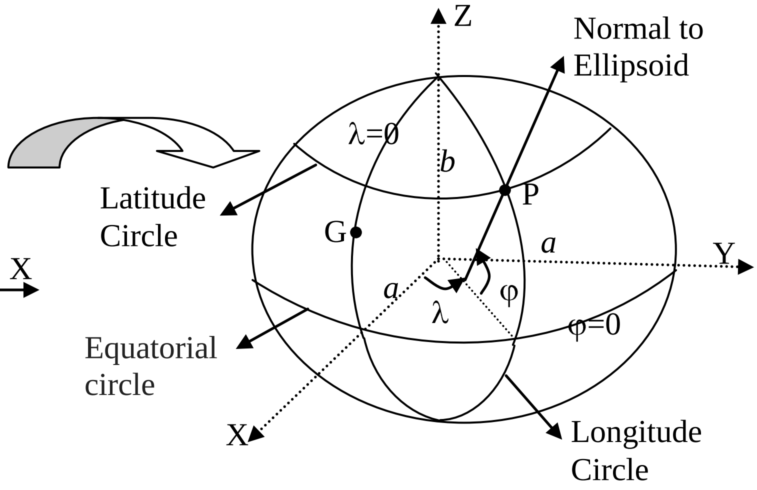

Figure-2 Rotational Elipsoid

$$\frac{x^2}{a^2}+\frac{z^2}{b^2}=1 \qquad (1)$$

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1 \qquad (2)$$

Ellipsoid

An ellipsoid is a closed quadric surface that is analogue of an ellipse. Ellipsoid has three different axes (a_x>a_y>b) as shown in Fig.3. Mathematical literature often uses “ellipsoid” in place of “Triaxial ellipsoid or general ellipsoid”. Scientific literature (particularly geodesy) often uses “ellipsoid” in place of “biaxial ellipsoid,rotational ellipsoid or ellipsoid revolution”. Older literature uses ‘spheroid’ in place of rotational ellipsoid. The standart equation of an ellipsoid centered at the origin of a cartesian coordinate system and aligned with the axes.

$$\frac{x^2}{a_x^2}+\frac{y^2}{a_y^2}+\frac{z^2}{b^2}=1 \qquad (3)$$

The following definitions will be used.

a_x = equatorial semimajor axis of the ellipsoid

a_y = equatorial semiminor axis of the ellipsoid

b = polar semi-minor axis of the ellipsoid

λ = geodetic longitude

φ = geodetic latitude

h = ellipsoid height

$e_x^2= (a_x^2-b^2)/a_x^2$ first polar eccentricity

$e_e^2= (a_x^2-a_y^2)/a_x^2$ first equatorial eccentricity

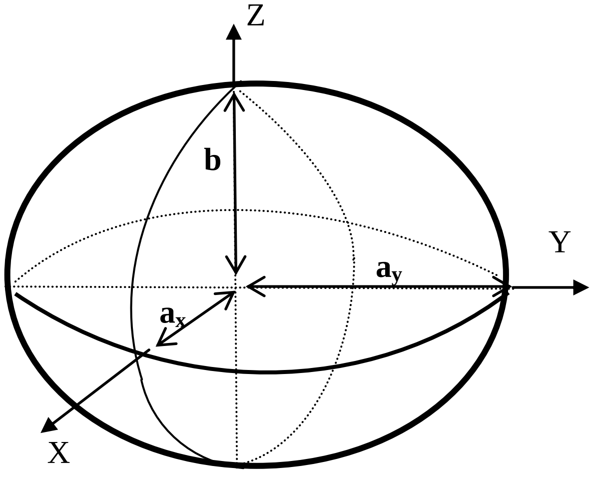

Figure-3. *Ellipsoid*

The differences between Rotational Elipsoid and Triaxial Ellipsoid:

-In Rotational ellipsoid has got two differents semi axes (a,b), whereas in Triaxial ellipsoid has got three differents semi axes (a_x,a_y,b)

-In Rotational ellipsoid ,all latitude circles (φ=constant) are circle geometry whereas in Triaxial ellipsoid, all latitude circles are ellips geometry. But one in any case,these lines are not plane lines

-In rotational ellipsoid ,all longitude circles (λ=constant) are the same ellips geometry, Ellips’s semi axis(a,b) equals to rotational ellipsoid. Whereas in Triaxial ellipsoid, all longitude circles are different ellips geometry. These ellips semi axis are (a_λ,b) but semi-major axis a_λ is changeable depending on longitude ($a_y \leq a_\lambda \leq a_x$)

$$a_\lambda = a_x \sqrt{\frac{\cos^2\lambda + (1-e_e^2)^2 \sin^2\lambda}{1-e_e^2 \sin^2\lambda}} \qquad (4)$$

λ meridian ellipse's semi major-axis In Rotational ellipsoid ,all surface's normal cuts to z-axis whereas in Triaxial ellipsoid, all surface's normal don't cuts the z-axis.

Ellipsoid equation (u,v) Gauss Parametric form

Cartesian coordinates from parametric coordinates

x = a_x cos u sin v (5.a)

y = a_y sin u sin v (5.b)

z = b cos v (5.c)

$-\pi/2 \le u \le \pi/2$, $-\pi \le v \le \pi$

Parametric coordinates from Cartesian coordinates

$$v = \arccos(\frac{z}{b}) \qquad (6.a)$$

$$u = \arctan(\frac{a_x.y}{a_y.x}) \qquad (6.b)$$

The parameter lines (u,v) and geodetic (planetographic) coordinates (φ, λ) are orthogonal on rotational ellipsoid but are not orthogonal on triaxial ellipsoid.

For a triaxial ellipsoid Jacobi employed**[5]** the ellipsoidal latitude and longitude (β,ω) defined by

$$x = a_x.\cos w \sqrt{\frac{a_x^2 - a_y^2 \sin^2\beta - b^2\cos^2\beta}{a_x^2 - b^2}} \qquad (7.a)$$

$$y = a_y.\cos\beta.\sin w \qquad (7.b)$$

$$z = b.\sin\beta \sqrt{\frac{a_x^2 \sin^2 w + a_y^2 \cos^2 w - b^2}{a_x^2 - b^2}} \qquad (7.c)$$

The parameter lines (β,ω) are orthogonal on triaxial ellipsoid. On the other hand, in the geodetic application,the geodetic coordinates(φ, λ) are used. Therefore there is a need for a transformation between the two sets of coordinates. Such as presented by **[11]** for the case of a biaxial ellipsoid.

Computing β,ω from x,y,z cartesian coordinates on triaxial ellipsoid: There is no direct formula. Presented method I suggest iteratively as shown below in the formula;

$$w_{i+1} = \arctan\left(\frac{a_x.y\sqrt{(p^2-1)\sin^2\beta_i + 1}}{a_y.x.\cos\beta_i}\right). \qquad (8.a)$$

(i=0,1,2,3,..)

$$\beta_{i+1} = \arctan\left(\frac{a_y.z.\sin w_i}{b.y.\sqrt{1-p^2.\cos^2 w_i}}\right). \qquad (8.b)$$

Where $p = \sqrt{\frac{a_x^2 - a_y^2}{a_x^2 - b^2}}$ (8.c)

For initial value $w_o = \lambda$ and $\beta_o = \varphi$

2.1. Fitting Ellipsoid

An ellipsoid is a closed quadric surface that is analogue of an ellipse. Ellipsoid has three different axes *(a_x>a_y>b)* in Fig.3. Mathematical literature often uses "ellipsoid" in place of "Triaxial ellipsoid or General ellipsoid". Scientific literature (particularly geodesy) often uses "ellipsoid" in place of "biaxial ellipsoid, rotational ellipsoid or ellipsoid revolution". Older literature uses 'spheroid' in place of rotational ellipsoid. The standard equation of an ellipsoid centered at the origin of a cartesian coordinate system and aligned with the axes is shown with this formula:

Although ellipsoid equation is quite simple and smooth but computations are quite difficult on the ellipsoid. The main reason for this difficulty is the lack of symmetry. Generally an ellipsoid is defined 9 parameters. These parameters are; 3 coordinates of center (X_o,Y_o,Z_o), 3 semi-axes (a_x,a_y,b) and 3 rotational angles (ε, ψ, ω) which represent rotations around *x*-,*y*- and *z*- axes respectively (Fig.2). These angles control the orientation of the ellipsoid**.** Similarity transformations with no scaling between (*xyz*) and (*XYZ*) see Appendix-A

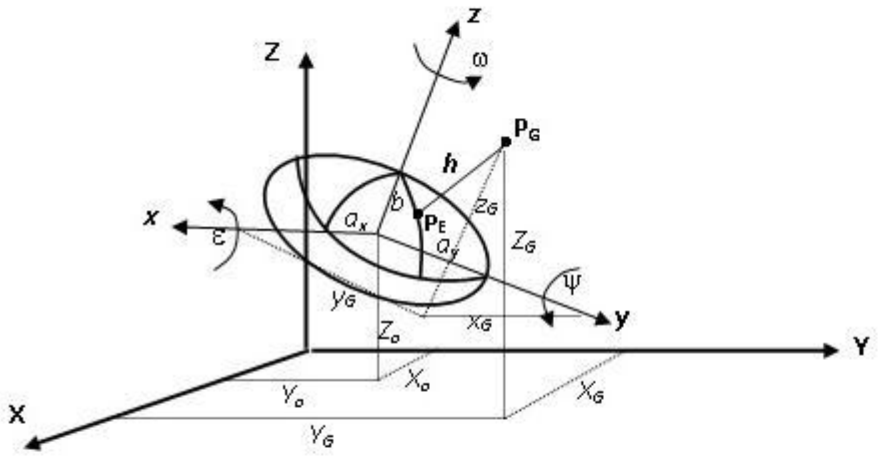

Figure 2. *Shifted - oriented ellipsoid*

For the solution of the fitting problem, the linear or linearized relationship between the given data points and unknown parameters (one equation per data points), is expressed by:

$$\Omega_{n,u} \,.\delta p_u = l_n \qquad (2)$$

Here, Ω is for design matrix, δp is shifted unknown parameters, l is measurements vector or data points,

Necessary conditions for this minimization problem to have a unique solution are that n >= 9 and the data points lie in general position (e.g., do not all lie is an elliptic plane).Throughout this paper, we assume that these conditions are satisfied.

u=9 : number of unknown parameter

n: number of given data point (or measurements)

f=n-u : degrees of freedom

-If $f = 0$ there is only one (exact) solution, algebraic solution

-If f <0 there is no solution. The solution can be found with based on the extra constraint

-If $f > 0$ commonly encountered situations. The given data points (or measurements), which is much than the required number cause discrepancy between data points and in this case, the solution is not unique. There is a over determined system. Because $n > u$, in other words number of equations great than number of unknowns.

The system of linear equations Eq. (2) must be solved. Therefore, this system must be consistent with the rank of design matrix and design matrix extended with constant terms, must be equal, so that *rank*(Ω) =*rank*(Ω:*l*); whereas, the system of Eq. (2) is inconsistent because shifted δx unknown parameters, not provided in Eq. (2), are not calculated. In this case, *rank*(Ω) $\leq u$. The extended matrix with l measurements *rank*(Ω:*l*) is generally more than *rank*(Ω) . There is no solution of inconsistent equations. In this case, only the approximate solution of the system can be derived. The equation system with approximate solution is calculated by adding ε residuals (or corrections) at the right side of Eq.(2).

$$\Omega_{n,u} .\delta p_u = l_n + \varepsilon_n \tag{3}$$

Depending on the choice of ε residuals vector, infinite solutions can be obtained. The unique solution should be made according to an estimator (objective function). For example, the LS always give a unique solution. Here, the question comes to mind what is should be the estimation method? On the convenience of the LS approach over the use of alternative norms should be sufficient.

3. ALGEBRAİC ELLİPSOİD FİTTİNG METHODS

The general equation of an ellipsoid is given as

$$A' x^2 + B' y^2 + C' z^2 + 2D' xy + 2E' xz + 2F' yz + 2G' x + 2H' y + 2I' z + K' = 0 \tag{4}$$

Since Eq.(4) is linear in the 9 ellipsoid parameters it is easy but meaningless to minimize the sum of the squares of the disclosures of Eq.(4) for given points obviously not on an ellipsoidal surface. This is the so called algebraic approach which lacks any justification as a best fitting method.

Eq.(4) contains ten parameters, In fact, nine of those ten parameters are independent. For example, all the coefficients in this equation multiply by (-1/K'), we get new equation which contains nine unknown parameters and its constant term will be equal "-1"

$$A x^2 + B y^2 + C z^2 + 2D xy + 2E xz + 2F yz + 2G x + 2H y + 2I z - 1 = 0 \tag{5}$$

This fitting algorithm, we need to check whether a fitted shape is an ellipsoid. In theory, the conditions that ensure a quadratic surface to be an ellipsoid have been well investigated and explicitly stated in analytic geometry textbooks. Since an ellipsoid can be degenerated into other kinds of elliptic quadrics, such as an elliptic paraboloid. Therefore a proper constraint must be added. (Li and Griffiths 2004) with the following definitions.

$i=A+B+C$

$j=AB+BC+AC-F^2-E^2-D^2$

However $4j-i^2 > 0$ is just a sufficient condition to guarantee that an equation of second degree in three variables represent an ellipsoid, but it is not necessary. In this paper, we assume that these conditions are satisfied.

Algebraic methods is a linear LS problem. It is solving directly easily. Given the data set ($(x,y,z)_i$, $i=1,2,\ldots,n$), the fitted ellipsoid by obtaining the solution in the LS sense of the linear algebraic equations.

$$\Omega . p = l \tag{6}$$

Where

Ω_{nxu} = Design matrix

$p_u = [\,A \;\; B \;\; C \;\; D \;\; E \;\; F \;\; G \;\; H \;\; I\,]^T$ unknown conic parameters

$l_n = [1 \;\; 1 \;\; 1 \ldots 1]^T$ unit vector : right side vector

ith row of the nx9 matrix Ω

$$[\, x_i^2 \;\; y_{i.}^2 \;\; z_i^2 \;\; 2x_i y_i \;\; 2x_i z_i \;\; 2y_i z_i \;\; 2x_i \;\; 2y_i \;\; 2z_i \,] \tag{7}$$

it is solved easily in the LS sense as below

$$p = (\Omega^T \Omega)^{-1} \Omega^T l \tag{8}$$

or it is solved easily as MATLAB expression:

$$p = [\, x.^2 \;\; y.^2 \;\; z.^2 \;\; 2x.y. \;\; 2x.z. \;\; 2y.z. \;\; 2x \;\; 2y \;\; 2z \,] \setminus l_n \tag{9}$$

If there is different of weight or correlation between given data points, so added P weight matrix in solution and then

$p = (\Omega^T P\, \Omega)^{-1}\, \Omega^T P l$ (10)

P $=K_{ll}^{-1}$ K_{ll} : *nxn* variance-covariance matrix of data points

Residual (or correction) vector computing as below

$\varepsilon = [\, x.^2 \;\; y.^2 \;\; z.^2 \;\; 2x.\,y. \;\; 2x.z. \;\; 2y.z. \;\; 2x. \;\; 2y. \;\; 2z.\,]\,.\, p - \; l_n$ (11)

LS optimization give us $[\varepsilon\varepsilon]$ =min.

Algebraic methods all have indisputable advantage of solving linear LS problem. The methods for his are well known and fast. However, it is intuitively unclear what it is we are minimizing geometrically Eq.(5) is often referred to as the "algebraic distance" to be minimized (Ray and Srivastava 2008). A geometric interpretation given by (Bookstein 1979) clearly demonstrates that algebraic methods neglect points far from the center. For best fitting of ellipsoid see (Bektas 2015)

4. COORDİNATE TRANSFORMATİON

Transformation from Cartesian coordinates (x, y, z) to geodetic coordinates (φ, λ, h) is one of the basic tasks in computational geodesy. Formulas for this transformation will be given in the next section.

4.1. Geodetic -Cartesian Coordinate Relationships (φ, λ, h) ⇒ (X, Y, Z)

The direct procedure is fairly simple and straightforward. Formulas for this transformation will be given below. The direct problem involves the computation of the X, Y and Z coordinates of a point given the latitude, longitude, and height. These relationships are well known and can be derived with the help of the relationships shown in Fig.4 . The important issue is to determine radius of curvature in the prime vertical (*v*) depending on longitude and latitude.

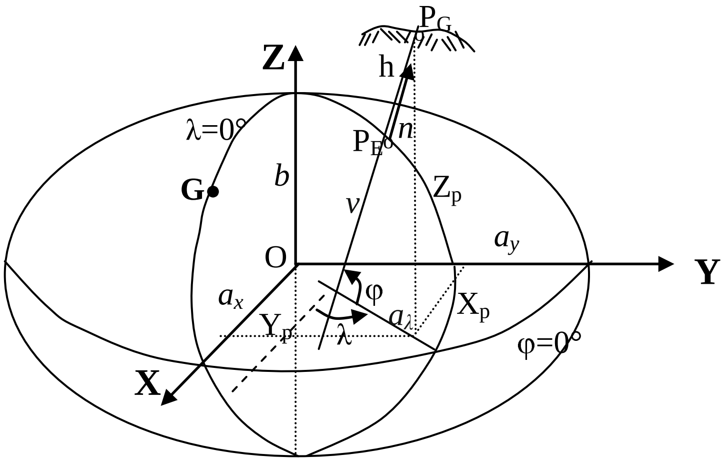

Figure-4. *(X,Y,Z) Cartesian (x,y,z) and Geodetic (φ, λ, h) coordinates on Triaxial ellipsoid*

The direct problem is solved using: Formulas relating geodetic and cartesian coordinates of a point P_E on a triaxial ellipsoid may be expressed **[3], [6]**

If point P_G outside/inside triaxial ellipsoid introducing ellipsoid height *h* in the below equations we will obtain general formulas relating to geodetic and cartesian coordinates of the form:

$x_G = (v+h) \cos \varphi \cos \lambda$ (9.a)

$y_G = [\, v\,(1 - e_e^2\,) + h] \cos \varphi \sin \lambda$ (9.b)

$z_G = [v\,(1 - e_x^2) + h] \sin \varphi$ (9.c)

$$v = \frac{a_x}{\sqrt{1 - e_x^2 \sin^2 \varphi - e_e^2 \cos^2 \varphi \sin^2 \lambda}}$$ radius of curvature in the prime vertical (9.d)

The following link can be used for calculations

http://www.mathworks.com/matlabcentral/fileexchange/46239-converter-geodetic-coordinates-to-cartesian-coordinates

4.2. Cartesian- Geodetic Coordinate Relationships (X, Y, Z) ⇒ (φ, λ, h)

The inverse association is not that easy, if point P_G outside/inside triaxial ellipsoid then there is no direct method. If point P_E is on the surface of triaxial ellipsoid then there is a direct solution. The direct procedure is fairly simple and straightforward. Generally, the problem of transformation cartesian to geodetic coordinates may be executed in two stages. The first step is to find the projection of an external point denoted as $P_G(x_G, y_G, z_G)$ in Fig.4 above you can see that the ellipsoid height is along the normal to this surface i.e. point P_E (x_E, y_E, z_E) **[3]**, **[6]** . For detailed information on this subject refer to **[1].** The following link can be used for the shortest distance and projection coordinates on triaxial ellipsoid.

After the first step is accomplished, founding P_E on the ellipsoid, the solution for φ, λ and h are found directly in the below formulas:

For Latitude,

$$\varphi = \arctan\left(\frac{(1-e_e^2)}{(1-e_x^2)}\frac{z_E}{\sqrt{(1-e_e^2)^2.x_E^2 + y_E^2}}\right) \tag{10.a}$$

For Longitude,

$$\lambda = \arctan\left(\frac{1}{(1-e_e^2)}\frac{y_E}{x_E}\right) \qquad X \geq 0$$

$$\lambda = \arctan\left(\frac{1}{(1-e_e^2)}\frac{y_E}{x_E}\right) + 180^o \qquad X < 0 \quad \text{ve} \quad Y \geq 0 \tag{10.b}$$

$$\lambda = \arctan\left(\frac{1}{(1-e_e^2)}\frac{y_E}{x_E}\right) - 180^o \qquad X < 0 \quad \text{ve} \quad Y < 0$$

For height;

If known coordinates of the projected point P_E

$$h = sign(z_G - z_E).sign(z_E)\sqrt{(x_E - x_G)^2 + (y_E - y_G)^2 + (z_E - z_G)^2} \tag{10.c}$$

or P_E coordinates not known,

$h = x_G / (\cos\varphi \cos\lambda) - v$

The following link can be used for calculations

http://www.mathworks.com/matlabcentral/fileexchange/46248-converter-cartesian-coordinates-to-geodetic-coordinates

5. ANALYSİNG CHANGE OF GEODETİC COORDİNATES

We have made some application to see how much difference is made using rotational ellipsoid instead of triaxial ellipsoid. In these application $\Delta a = a_y - a_x$ =70m is based. Earth's geometrical parameters a_x= 6378388.0000 a_y = 6378318.0000 b= 6356911.9461

The results are shown in Table 1.

Table-1. *Changes latitude and ellipsoidal height depending latitude ($\lambda=90^o$)*

Latitude φ	0^o	15	30	45	60	75	90
$\Delta\varphi = \varphi_2 - \varphi_3$	0"	1.135	1.965	**2.267**	1.962	1.132	0
$\Delta h = h_2 - h_3$	**70m**	65.325	52.544	35.058	17.544	4.704	0

Subscribe $_{2,3}$ denotes results of rotational and triaxial ellipsoid,respectively.

6. DISCUSSİON AND CONCLUSİON

In this paper we study on the computational differences in results and use of rotational ellipsoid instead of triaxial elipsoid, Increase of the equatorial semiaxis difference (Δa) will cause big differences in the calculation results.

We see that maximum changes are on the $\pm 90^o$ longitude. According to our research as a result analyzing changes in latitude, maximum change is $\Delta\varphi_{max}$ = 2.267" on the $\pm 45^o$ latitude are as shown in Table 1. If longitude closer to 90^o differences are increasing. This difference of latitude corresponds to 70 meters(=Δa) offsets on the surface of ellipsoid.

In longitude maximum change is $\Delta\lambda_{max}$ = -2.263" on the $\pm 45^o$ degree latitude independently of latitude.

In ellipsoidal height maximum change is on the equator 0^o and $\pm 90^o$ longitude $\Delta h_{max}=\Delta a$ =+70m is as shown in Table 1. The differences of ellipsoidal height is close to zero when approaching the poles.

It is seen from the studies that maximum change of latitude and ellipsoidal height corresponds to 70 meters(=Δa) offsets. Even if differences in equatorial semiaxis Δa=10 meters. This leads to 10 meters offsets horizontal and vertical on the surface of the ellipsoid. This offsets can not be ignored. These differences are important in the geodesic sense. A triaxial ellipsoid is fitting better to geoid than the rotational ellipsoid. Geodetic applications in the near future will be used for triaxial ellipsoid or more advanced surfaces or multiple axis ellipsoid as expected. As a result to adapt better to the Earth and we will be getting smaller geoid undulations.

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Appendix

Similarity transformations with no scaling between (xyz) and (XYZ) (Fig.2).

R_{3x3}-rotation matrix is obtained from the rotational angles

$$R = \begin{bmatrix} \cos\psi\cos\omega & \cos\varepsilon\sin\omega + \sin\varepsilon\sin\psi\cos\omega & \sin\varepsilon\sin\omega - \cos\varepsilon\sin\psi\cos\omega \\ -\cos\psi\sin\omega & \cos\varepsilon\cos\omega - \sin\varepsilon\sin\psi\sin\omega & \sin\varepsilon\cos\omega + \cos\varepsilon\sin\psi\sin\omega \\ \sin\psi & -\sin\varepsilon\cos\psi & \cos\varepsilon\cos\psi \end{bmatrix}$$

T_{4x4}- transformation matrix is obtained from the R_{3x3} rotational matrix and the shifted parameters (X_0,Y_0,Z_0)

$$\mathbf{T_{4x4}} = \begin{bmatrix} & & & X_0 \\ & R_{3x3} & & Y_0 \\ & & & Z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} = \mathbf{T}^{-1} \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix}$$

(xyz) ➔ (XYZ) (XYZ)➔ (xyz)