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Performance Evaluation of Nakagami fading Channel for Frequency Shift Keying (FSK)

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Abstract: *In digital communication performance of frequency shift keying, the average SEP of orthogonal M-FSK evolves the (M-1) power of the Gaussian Q- function. This is nevertheless possible to obtain simple to evaluate asymptotically tight upper bounds on the average error probability performance of 2-ary and 4-ary FSK on the Rayleigh and Nakagami fading channels*.

Keywords: *FSK, Fading Channels, SEP and AWGN.*

1. Introduction

In this paper, the performance of these frequency shift keying communication systems over generalized fading channels. Wherever possible, we shall again make use of the desired forms rather than the classical representations of the mathematical functions the treatment here will be quit brief since indeed the entire machinery that allows determining the results has by this time been developed completely. Thus, for the most part we shall merely present the final results except for the few situations where further developed is warranted.

When fading is present, the received carrier amplitude, A_c, is attenuated by the fading amplitude, α, which is a random variable (RV) with mean-square value $\overline{\alpha}^2 = \Omega$ and probability density function (PDF) dependent on the nature of the fading channel. Equivalently, the received instantaneous signal power is attenuated by α^2, and thus it is appropriate to define the instantaneous SNR per bit by $\gamma \triangleq \alpha^2 E_b / N_o$ and the average SNR per bit by $\overline{\gamma} \triangleq \alpha^2 E_b / N_o = \Omega E_b / No.$ As such, conditioned on the fading, the BEP of FSK the modulation considered in this chapter we obtained by replacing E_b/N_o by γ in the expression for AWGN performance. Denoting this conditional BEP by $P_b(E;\gamma)$ the average BEP in the presence of fading is obtained from

$$P_b(E) = \int_0^\infty P_b(E;\gamma) p_\gamma(\gamma) d\gamma \tag{1}$$

Where $p_\gamma(\gamma)$ is the PDF of the instantaneous SNR. On the other hand, if one is interested in the average SEP, the same relation as equs. (1) applies using, instead, the conditional SEP in the integrand, which is obtained from the AWGN result with E_s/N_o replaced by $\gamma \log_2 M$. Our goal in this paper is to evaluate by eque. (1) for FSK modulation and detection schemes over fading channel model.

2. M-ary Frequency-Shift-Keying

Consider first the case of orthogonal signaling using M-FSK modulation described by the signal occurs when *f* (t) takes on equiprobable values ξ= (2i-1 – M) $\Delta\int / 2, i = 1,2,\ldots,M,$ in each symbol interval T_s where the frequency spacing $\Delta\int$ is related to the frequency modulation index h by h $=\Delta f T_s$. As such, f (t) is modeled as a random pulse stream, that is,

$$f(t) = \sum_{n=-\infty}^{\infty} \int_n p(t - nTs) \tag{2}$$

Where f_n is the information frequency in the nth symbol interval $nT_s \leq t \leq (n+1)T_s$ ranging over the set of M possible values ξ_i as above, and $p\ (t)$ is again a unit amplitude rectangular pulse of duration T_s seconds. Thus the complex signal transmitted in the nth symbol interval is

$$\tilde{s}(t) = A_c e^{j[2\pi(fct + fn(t-nTs)) + \theta c]} \tag{3}$$

The complex baseband modulation $\hat{S}\ (t) = A_c e^{jfn(t-nTs)}$ is not constant over this same interval but rather has a sinusoidal variation. After demodulating with the complex conjugate of $\hat{C}_r$ (t) at the receiver, we obtain

$$\tilde{x}(t) = A_c e^{j2\pi fn(t-nTs)} + \tilde{N}(t) \tag{4}$$

Multiplying (4) by the set of harmonics $e^{-j2\pi\xi k(t-nTs)}$, k = 1, 2, ...M, and the passing each resulting signal through an I&D produces the decision variables (Fig 1)

$$\tilde{y}nk = A_c \int_{nT_s}^{(n+1)T_s} e^{j2\pi(fn-\xi k)(t-nTs)} dt + \tilde{N}_{nk},\ k = 1, 2, ..., M,$$

$$\tilde{N}nk = \int_{nT_s}^{(n+1)T_s} e^{-j2\pi\xi k(t-nTs)}\ \tilde{N}(t)dt \tag{5}$$

from which a decision corresponding to the largest Re $\{\tilde{y}nk\}$ is made on the information frequency transmitted in the nth signaling interval.

For orthogonal signaling wherein the cross-correction Re

$$\left\{ \int_{nTs}^{(n+1)Ts} \tilde{s}k(t)\ \tilde{s}_l^{*}(t)dt \right\} = 0,\ k \neq 1,$$

the frequency spacing is chosen such that Δf=N/2T_s with N integer. If, the transmitted frequency fn is equal to ξ_l= (2l-1 - M)Δf/2, then (5) can be expressed as

$$\tilde{y}_{nk} = A_c T_c e^{j\pi(l-k)n/2} \frac{\sin[\pi(l-k)N/2]}{\pi(l-k)N/2} + \tilde{N}_{nk}\ k = 1, 2, ..., M,$$

$$\tilde{N}_{nk} = \int_{o}^{T_s} e^{-j\pi(2k-1-M)Nt/2Ts}\ \tilde{N}(t + nTs)dt \tag{6}$$

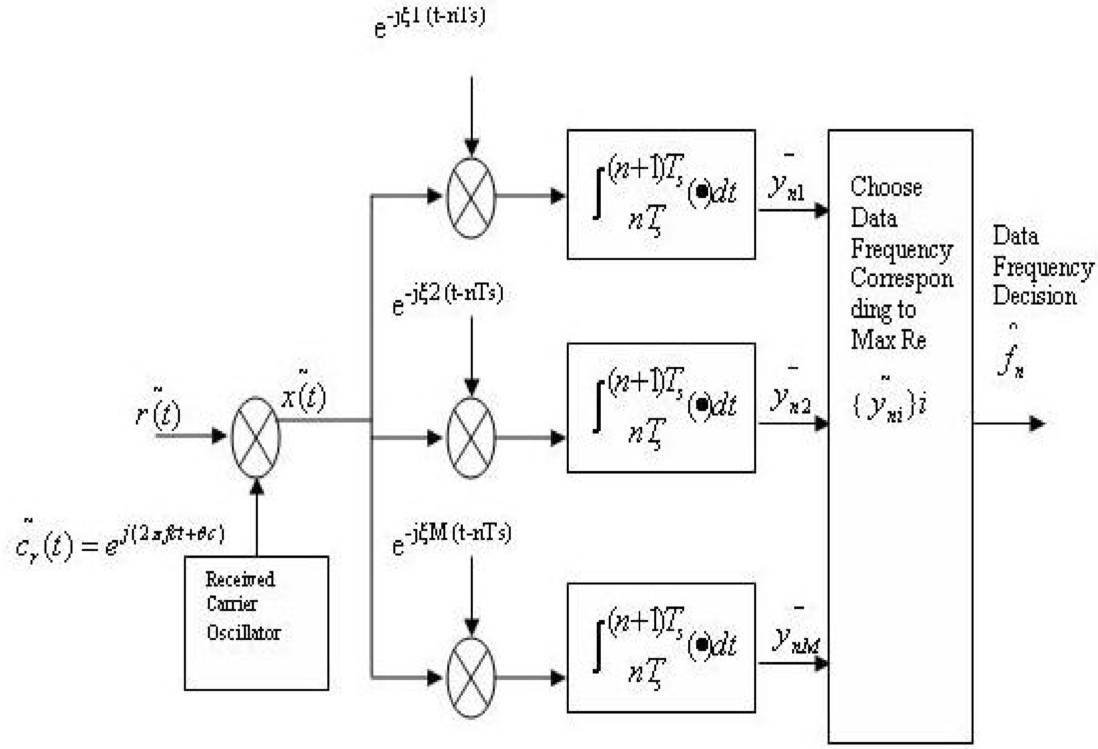

Figure 1. Complex form of optimum receiver for ideal coherent detection of M-FSK over the AWGN

Or, taking the real part,

$$\text{Re}\,\{\bar{y}_{nk}\} = A_c T_s \frac{\sin[\pi(l-k)N]}{\pi(1-k)N} + \text{Re}\{\tilde{N}_{nk}\} \qquad k = 1, 2 \ldots M \tag{7}$$

Thus we observe that for orthogonal M-FSK, only one variable has a nonzero mean: the one corresponding to the transmitted frequency. That is,

$$\overline{\text{Re}\{\tilde{y}_{nl}\}} = A_c T_{s,}\ \overline{\text{Re}\{\tilde{y}_{nk}\}} = 0, k \neq 1 \tag{8}$$

A popular special case of M-FSK modulation is binary FSK (BFSK), which corresponds to M=2. In addition to orthogonal signaling (zero cross-correlation), it is possible to choose the modulation index so as to achieve the minimum cross-correlation that results in the minimum error probability. Since for arbitrary Δf we have

$$\text{Re}\left\{\int_{nT_b}^{(n+1)T_b} \tilde{s}_1(t)\, \tilde{s}_2^*(t)\,dt\right\} = \text{Re}\left\{A_c^2 \int_o^{T_b} e^{-j2\pi\Delta ft} dt\right\}$$

$$= A_c^2 T_b \frac{\sin 2\pi\Delta f T_b}{2\pi\Delta f T_b} \tag{9}$$

The minimum of this cross-correlation is achieved when $h = \Delta f T_b = 0.715$ [1], which results in a minimum normalized cross-correlation value

$$\rho \triangleq \frac{\mathrm{Re}\left\{ \int_{nT_b}^{(n+1)T_b} \tilde{s}1(t)\, \tilde{s}_2^{*}(t)\,dt \right\}}{\int_{nT_b}^{(n+1)T_b} \left| \tilde{s}1(t) \right|^2 dt} = \frac{\mathrm{Re}\left\{ \int_{nT_b}^{(n+1)T_b} \tilde{s}1(t)\, \tilde{s}_2^{*}(t)\,dt \right\}}{\int_{nT_b}^{(n+1)T_b} \left| \tilde{s}2(t) \right|^2 dt}$$

$$= \frac{\sin 2\pi\Delta f T_b}{2\pi\Delta f T_b}\Big|\Delta f T_b = 0.715 = -0.217 \simeq -\frac{2}{3\pi} \tag{10}$$

Now, we consider first the case of orthogonal signaling using M-FSK modulation already described. and the receiver of Fig.1. Assuming that the transmitted frequency in the *n*th symbol interval, $f_{n,}$ is equal to ξ_l = (2*l* - 1 - M)$\Delta f/2$, the real parts of the integrate-and-dump (I & D) outputs, $\tilde{y}_{nk,} k = 1, 2...., M$ as given by eque.(10) are independent, identically disturbed (i.i.d.). Gaussian random variable with means as in equ.(10) and variance $\sigma_n^2 = N_o T_s / 2.$ The probability of a correct symbol decision is the probability that all $\mathrm{Re}\{\ \tilde{y}_{nk}\}$, k = ≠ *l*, are less than $\mathrm{Re}\{\ \tilde{y}_{nl}\}$. Thus, letting A_c = $\sqrt{E_s / T_s}$ and denoting $\mathrm{Re}\{\ \tilde{y}_{nl}\}$ by $z_{nk,}$

3. Symbol Error Probability

The probability of symbol error is given by.

$$P_s(E) = 1 - \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z_{nl}} \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{z^2_{nk}}{2\sigma_n^2} \right) dz_{nk} \right]^{M-1}$$

$$\times \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left[-\frac{\left(z_{nk} - \sqrt{E_s T_s} \right)^2}{2\sigma_n^2} \right] dz_{nl} \tag{11}$$

Or in term of the Gaussian Q-function,

$$P_s(E) = 1 - \int_{-\infty}^{\infty} \left[Q\left(-q - \sqrt{\frac{2E_s}{N_o}} \right) \right]^{M-1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{q^2}{2} \right) dq \tag{12}$$

The corresponding bit error probability is given by

$$P_b(E) = \frac{2^{k-1}}{2^{k-1}} P_s(E), k = \log_2 M \qquad (13)$$

Unfortunately, for arbitrary M, equ.(12) cannot be put in the desired from by using the form of the Gaussian Q-function. The special case of binary orthogonal FSK (M=2), however, does have a simple from, namely,

$$P_b(E)= Q\left(\sqrt{\frac{E_b}{N_o}}\right) \qquad (14)$$

This can put in the desired form,

$$P_b(E) = \frac{1}{\pi}\int_o^{\pi/2} \exp\left(-\frac{E_b}{2N_o}\frac{1}{\sin^2\theta}\right)d\theta \qquad (15)$$

Another M-FSK case whose error probability performance can be put into the desired from corresponds to binary non-orthogonal FSK with cross-correlation given by equ.(8). In particular, the BEP for such a modulation is given,

$$P_b(E) = Q\left(\sqrt{\frac{E_b(1-\sin 2\pi h/22\pi h)}{No}}\right)$$

$$= \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{E_b}{2No} - \frac{(1-\sin 2\pi h/2\pi h)}{\sin^2\theta}\right)d\theta \qquad (16)$$

where, as before, $h=\Delta f T_b$ is the frequency-modulation index. The minimum BEP is achieved when $h = 0.714$ (the value of h that maximizes the argument of the Gaussian Q-function), resulting in

$$P_b(E) = Q\left(\sqrt{\frac{E_b(1.217)}{No}}\right) = \frac{1}{\pi}\int_o^{\pi/2} \exp\left(-\frac{E_b 1.217)}{2N_o \sin^2\theta}\right)d\theta \qquad (17)$$

which is often approximately by

$$P_b(E) = Q\left(\sqrt{\frac{E_b(1+2/3\pi)}{No}}\right) = \frac{1}{\pi}\int_o^{\pi/2} \exp\left(-\frac{E_b 1+2/3\pi}{2N_o \sin^2\theta}\right)d\theta \qquad (18)$$

In above, we observed that the expression (12) for the average SEP of orthogonal M-FSK involves the (M-1) st power of the Gaussian Q-function. Since for M arbitrary an alternative from is not available for $Q^{M-1}(x)$, (12) cannot be put in the desired from to allow simple evaluation of the average SEP on the generalized fading channel. Despite this consequence, however, it is nevertheless possible to obtain simple to evaluate, asymptotically tight upper bounds on the average error probability performance of 4-ary FSK on the Rayleigh and Nakagami-m fading channels, as we shall show shortly. For the special case of binary FSK (M=2), we can use the desired from in (15) (from orthogonal signals) or (16) (for nonorthogonal signals) to allow simple exact evaluation of average BEP on the generalized fading channel. Before moving on to the more difficult 4-ary FSK case, we first quickly dispense with the results for binary FSK since these fellow immediately from the integrals developed or equivalently from the results obtained previously for binary AM and BPSK, replacing γ by γ /2 for orthogonal BFSK and by (γ/2) [1-(sin2πh)/2πh] fro nonorthogonal BFSK. For example, for Rayleigh fading the average BEP of orthogonal BFSK is given by

$$p_b(E) = \frac{1}{2}\left(1 - \sqrt{\frac{\bar{\gamma}/2}{1+\bar{\gamma}/2}}\right) \tag{19}$$

Whereas for Nakagami-m fading the analogous results are

$$p_b(E) = 2\left[1 - \mu \sum_{k=0}^{m-1} \binom{2k}{k} \left(\frac{1-\mu^2}{4}\right)^k\right], \mu \overset{\Delta}{=} \sqrt{\frac{\bar{\gamma}/2}{m+\bar{\gamma}/2}}, \ m \text{ integer} \tag{20}$$

And

$$P_b(E) = \frac{1}{2\sqrt{\pi}} \frac{\sqrt{\gamma/2m}}{(1+\gamma/2m)^{m+1/2}} \frac{\Gamma(m+\frac{1}{2}}{\Gamma(m+1)}$$

$$\times_2 F_1\left(1, m+\frac{1}{2}; m+1; \frac{1}{1+\bar{\gamma}/2m}\right), \ m \text{ no integer} \tag{21}$$

For M-ary orthogonal FSK, the average SEP on the AWGN can be obtained from emu. (12) As

$$P_s(E) = 1 - \int_{-\infty}^{\infty} \left[Q\left(-q - \sqrt{\frac{2E_s}{N_o}}\right)\right]^{M-1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{q^2}{2}\right) dq$$

$$= \int_{-\infty}^{\infty} \left\{1 - \left[1 - Q\left(q + \sqrt{\frac{2E_s}{N_o}}\right)\right]^{M-1}\right\} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{q^2}{2}\right) dq$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left\{1 - \left[1 - Q\left(\sqrt{2}\left(u + \sqrt{\frac{E_s}{N_o}}\right)\right)\right]^{M-1}\right\} \frac{1}{\sqrt{2\pi}} \exp\left(-u^2\right) du \tag{22}$$

and the corresponding BEP is obtained from equ.(22) using eque.(13). The most straightforward way of numerically evaluating eque.(22) (and therefore the BEP derived from it) is to apply Gauss-Hemite quadrature, resulting in

$$P_s(E) \square \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N_p} w_n \left\{1 - \left[1 - Q\left(\sqrt{2}\left(\chi_n + \sqrt{Es/No}\right)\right)\right]^{M-1}\right\} \tag{23}$$

where $\{x_n; n=1, 2, \ldots N_p\}$ are the zeros of the Hermite polynomial of order N_p and w_n are the associated weigh factors [2, Table 25.10]. A value of $N_p=20$ is typically sufficient for excellent accuracy.

When slow fading is present, the average symbol error probability is obtained from (22) or (23) by first replacing E_s/N_o with $\gamma=\alpha^2 E_s/N_o$ and then averaging over the PDF of γ, that is,

$$P_s(E) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{1-[1-Q(y)]^{M-1}\}$$

$$\times \int_{o}^{\infty} \exp\left(-\frac{(y-\sqrt{2\gamma})^2}{2}\right) p_\gamma(\gamma) d\gamma dy \tag{24}$$

Or approximately

$$P_s(E) \square \frac{1}{\sqrt{\pi}} \sum_{n=1}^{N_p} w_n \left\{1-\int_0^\infty [1-Q(\sqrt{2}(\chi_n+\sqrt{\gamma})\ )\ ]^{M-1} p_\gamma\left(\gamma\ \right) d_\gamma\right\} \tag{25}$$

Numerical evaluation of equ.(24) and the associated bit error probability for Rayleigh and Nakagami-m fading channels is computationally intensive. Equation (25) does yield numerical value; however, its evaluation is very time consuming, especially for large value of m. Thus, tight upper bounds on the result in equ.(24) which are simple to use and evaluation numerically are highly desirable.

Using Jensen's inquality [3], Hughes [4] derived a simple bound on the AWGN performance in equ. (22). In particular, it was shown that

$$P_s(E) \le 1-\left[1-Q\left(\sqrt{\frac{E_s}{No}}\right)\right]^{M-1} \tag{26}$$

This is tighter than the more common union upper bound [15, Eq. (4.97)]

$$P_s(E) \le (M-1) Q\left(\sqrt{\frac{E_s}{No}}\right) \tag{27}$$

Evaluation of an upper bound on average error probability for the fading channel by averaging the right-hand side of equ. (26) (with E_s/N_o replaced by γ_s) over the PDF of γ_s and using the conventional from for the Gaussian probability integral is still computationally intensive. Using the alternative forms of the Gaussian Q-function and its square, it is possible to simplify the evaluation of this upper bound on performance.

We begin by applying a binomial expansion to the Hughes bound of equ.(26), which when averaged over the fading PDF results in

$$P_s(E) \le \sum_{k=1}^{M-1} (-1)^{k+1} \binom{M-1}{k} l_k \tag{28}$$

Where

$$I_k \square \int_0^\infty Q^k(\sqrt{\gamma_s}) p\gamma_s(\gamma_s) d\gamma_{s,}\ \ k = 1, 2... M\text{-}1 \tag{29}$$

The result based on the union upper bound would simply be the first term k=1 of equ.(28). and the integral in equ.(29) can be evaluated for M=4 (k=1, 2, 3) either in closed form or in the form of a single integral with finite limits and an integrand composed of elementary functions. The results appear and are summarized here as follows:

$$I_1 = [P(c)]^m \sum_{k=0}^{m-1} \binom{m-1+k}{k} [1-P(c)]^k,$$

$$P(c) \triangleq \frac{1}{2}\left(1-\sqrt{\frac{c}{1+c}}\right), c \triangleq \frac{\bar{\gamma}_s}{2m} \tag{30}$$

$$I_2 = \frac{1}{4} - \frac{1}{\pi}\sqrt{\frac{c}{1+c}}\left\{\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{c}{1+c}}\right)\right\} \sum_{k=0}^{m-1} \binom{2k}{k} \frac{1}{[4(1+c)]^k}$$

$$-\sin\left(\tan^{-1}\sqrt{\frac{c}{1+c}}\right) \sum_{k=1}^{m-1} \sum_{i=1}^{k} \frac{T_{ik}}{(1+c)^k} \left[\cos\left(\tan^{-1}\sqrt{\frac{c}{1+c}}\right)\right]^{2(k-i)+1} \}$$

$$T_{ik} \triangleq \frac{\binom{2k}{k}}{\binom{2(k-i)}{k-i} 4^i [2(k-i)+1]}, \; c \triangleq \frac{\bar{\gamma}_s}{2m} \tag{31}$$

And

$$I_3 = \frac{1}{\pi}\int_{o}^{\pi/4} \left(\frac{2}{\bar{\gamma}} c(\varphi)\right)^m [P(c(\phi))]^m \sum_{k=0}^{m-1} \binom{m-1+k}{k} [1-P(c(\phi))]^k \, d\phi,$$

$$c(\phi) \triangleq \frac{\bar{\gamma}_s}{2}\left(\frac{\sin^2\phi}{\sin^2\phi + \bar{\gamma}/2}\right) \tag{32}$$

4. Result Analysis

In fig.-2, the graph show the curves for average bit error probability versus average bit SNR for 4-ary orthogonal signaling over the Nakagami-m channel, the special case of m=1 and m=2 corresponding to the Rayleigh channel. For each value of m, three curves are calculated. The first is the exact obtained (with much computational power and time) by averaging equ.(25) over the PDF in

$$P_\gamma(\gamma) = m^m \gamma^{m-1} \exp\left(-m\gamma / \bar{\gamma}\right), \qquad \gamma \geq 0$$

The second is the Hughes upper bound obtained from equ.(28) together with equ.(30), (31), and (32). Finally, the third is the union upper bound obtained from the first term of (25) together with (32).In fig.-3,The curves labeled m=4 and m=∞.We observe that as m increases (the amount of fading decreases) the three results are asymptotically equal to each other. For Rayleigh fading (the smallest integer value of m) we see the most disparity between the three, with the Hughes bound falling approximately midway between the exact result and the union upper bound. More specifically, the “averaged” Hughes bound is 1 dB tighter than the union bound for high-average bit SNR values. As m increases, the differences between the Hughes bound and the exact results is at worst less than a few tenths of 1 dB over a wide range of average bit SNR’s. Hence, for high values of m, we can conclude that it is accurate to use the former as a prediction of true system performance, with the advantage that the numerical results can be obtained instantaneously.

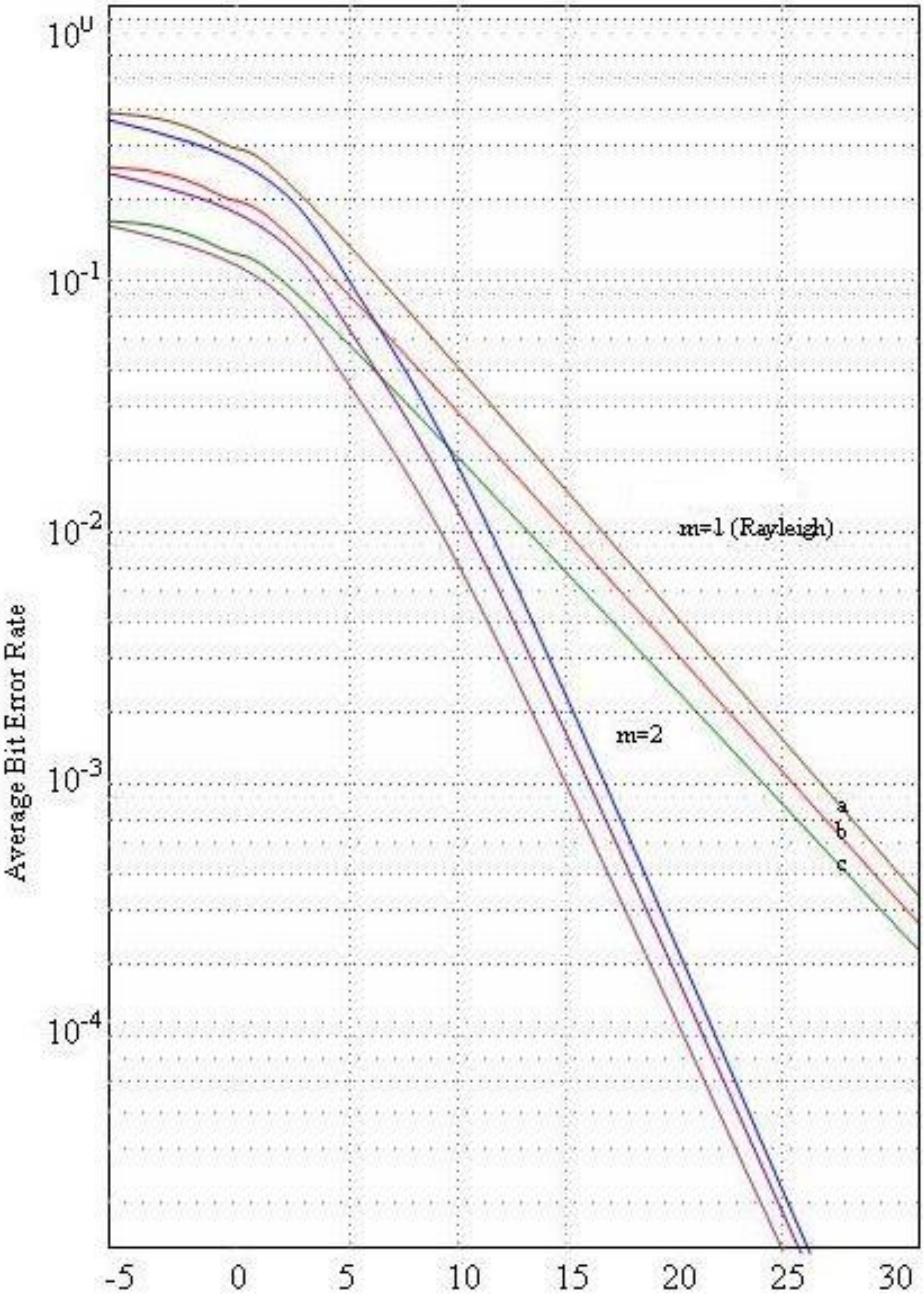

Figure 2 Average SNR per Bit [dB]
Average BEp of 4-ary orthogonal signals over Nakagami-m channel versus the average SNR per bit (a) union bound: (b) = Hughes bound: (c) exact result, for m=1 and 2.

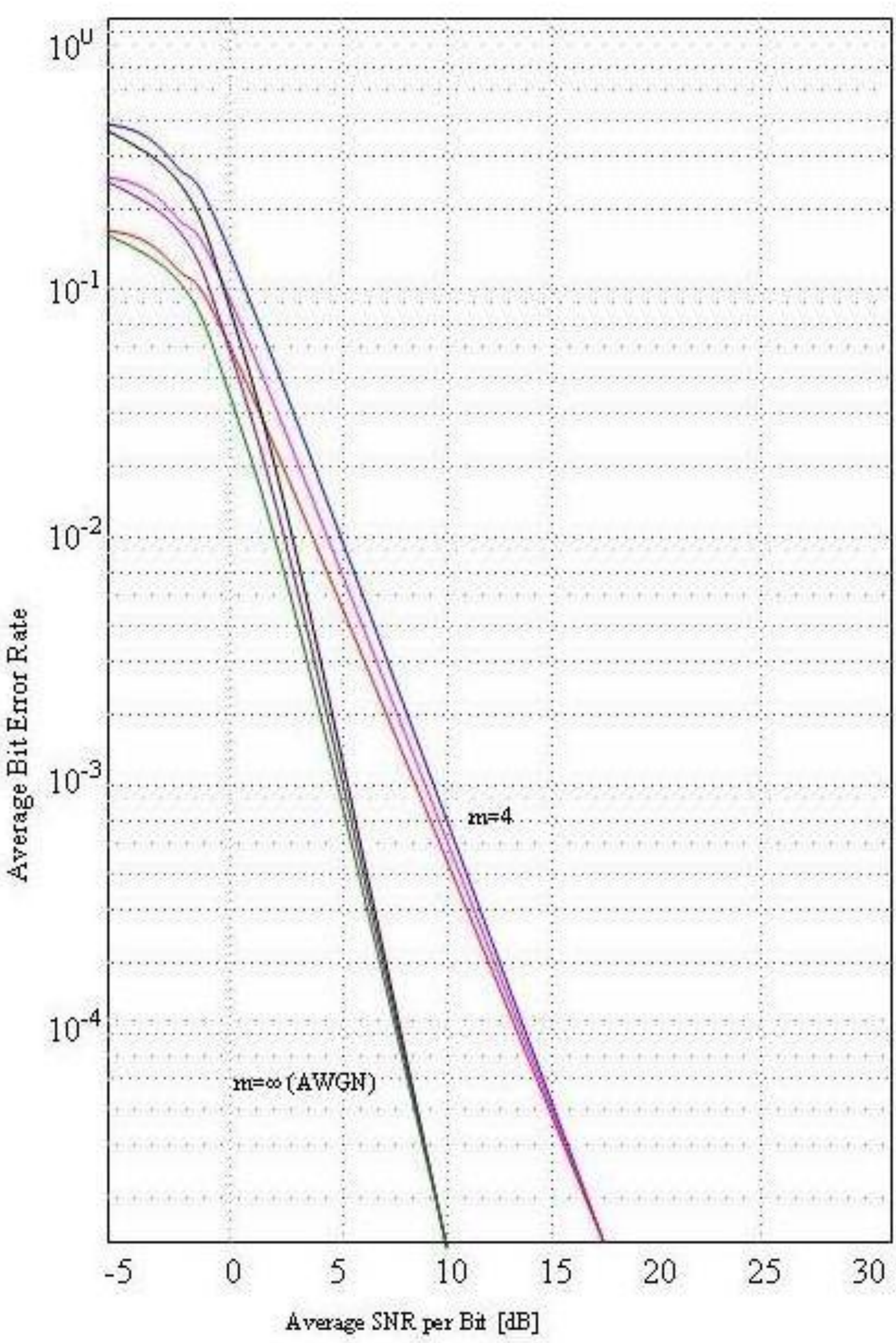

Figure 3- Average SNR per Bit [dB]
Average BEp of 4-ary orthogonal signals over Nakagami-m channel versus the average SNR per bit (a) union bound: (b) = Hughes bound: (c) exact result, for m=4 and ∞.

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