International Journal of Advanced Research in Physical Science (IJARPS)
Volume 6, Issue 12, 2019, PP 1-5
ISSN No. (Online) 2349-7882
www.arcjournals.org

Gauge Theory's Expansion in the Electro-Magnetic Field

Sangwha-Yi*

Department of Math , Taejon University 300-716 , South Korea

***Corresponding Author: Sangwha-Yi,** Department of Math, Taejon University 300-716, South Korea*

Abstract: *In the special relativity theory, we study the gauge theory in the electro-magnetic field theory. Using that the Electro-magnetic potential is 4-vector, we treat the invariant potential. Electro-magnetic field theory's the gauge theory is expanded*

Keywords: *electro-magnetic potential, 4-vecter, the gauge theory*

PACS Number: *003.30,41.20*

1. Introduction

In the special relativity theory, electro-magnetic potential $(\phi,\vec{A})$ is 4-vector likely time-space

$$\vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{c\partial t}\ ,\ \vec{B}=\vec{\nabla}\times\vec{A} \tag{1}$$

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2)A^\alpha=\frac{4\pi}{c}j^\alpha=\frac{4\pi}{c}\rho_0\frac{dx^\alpha}{d\tau} \tag{2}$$

$$\frac{1}{c}\frac{\partial}{\partial t}=\gamma(\frac{1}{c}\frac{\partial}{\partial t'}-\frac{\vec{v}}{c}\cdot\vec{\nabla}'),\vec{\nabla}=\vec{\nabla}'-\gamma\frac{\vec{v}}{c}\frac{1}{c}\frac{\partial}{\partial t'}-(1-\gamma)\frac{\vec{v}}{v^2}\cdot\vec{\nabla}'\vec{v} \tag{3}$$

$$ct=\gamma(ct'+\frac{\vec{v}}{c}\cdot\vec{x}'),\ \vec{x}=\vec{x}'+\gamma\frac{\vec{v}}{c}ct'-(1-\gamma)\frac{\vec{v}\cdot\vec{x}'}{v^2}\vec{v} \tag{4}$$

$$\phi=\gamma(\phi'+\frac{\vec{v}}{c}\cdot\vec{A}') \tag{5}$$

$$\vec{A}=\vec{A}'+\gamma\frac{\vec{v}}{c}\phi'-(1-\gamma)\frac{\vec{v}\cdot\vec{A}'}{v^2}\vec{v},\gamma=1/\sqrt{1-\frac{v^2}{c^2}} \tag{6}$$

2. Gauge Transformation

Gauge transformation of electro-magnetic potential $(\phi,\vec{A})$ of an inertial coordinate system S and electro-magnetic potential $(\phi',\vec{A}')$ of the other inertial coordinate system S'' is

$$\phi\to\phi-\frac{1}{c}\frac{\partial\Lambda}{\partial t},\ \vec{A}\to\vec{A}+\vec{\nabla}\Lambda$$

Λ is a function of S (7)

$$\phi'\to\phi'-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'},\ \vec{A}'\to\vec{A}'+\vec{\nabla}'\Lambda'$$

Λ' is a function of S' (8)

Therefore, Eq (5), Eq (6) are used by Eq (7) and Eq (8)

$$\phi-\frac{1}{c}\frac{\partial\Lambda}{\partial t}=\gamma\{\phi'-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'}+\frac{\vec{v}}{c}\cdot(\vec{A}'+\vec{\nabla}'\Lambda')\}$$

$$=\gamma\{\phi'+\frac{\vec{v}}{c}\cdot\vec{A}'\}+\gamma\{-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'}+\frac{\vec{v}}{c}\cdot\vec{\nabla}'\Lambda'\} \tag{9}$$

$$\vec{A}+\vec{\nabla}\Lambda=\vec{A}'+\vec{\nabla}'\Lambda'+\gamma\frac{\vec{v}}{c}(\phi'-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'})-(1-\gamma)\frac{\vec{v}}{v^2}\cdot(\vec{A}'+\vec{\nabla}'\Lambda')\vec{v}$$

$$=\{\vec{A}'+\gamma\frac{\vec{v}}{c}\phi'-(1-\gamma)\frac{\vec{v}}{v^2}\cdot\vec{A}'\vec{v}\}+\{\vec{\nabla}'\Lambda'+\gamma\frac{\vec{v}}{c}(-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'})-(1-\gamma)\frac{\vec{v}}{v^2}\cdot(\vec{\nabla}'\Lambda')\vec{v} \tag{10}$$

In this time, if Eq (9) ,Eq(10) are used by Eq (5), Eq(6) , the gauge function Λ and Λ''s relation is

$$-\frac{1}{c}\frac{\partial\Lambda}{\partial t}=\gamma(-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'}+\frac{\vec{v}}{c}\cdot\vec{\nabla}'\Lambda')=\gamma(-\frac{1}{c}\frac{\partial}{\partial t'}+\frac{\vec{v}}{c}\cdot\vec{\nabla}')\Lambda' \tag{11}$$

$$\vec{\nabla}\Lambda=\vec{\nabla}'\Lambda'+\gamma\frac{\vec{v}}{c}(-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'})-(1-\gamma)\frac{\vec{v}}{v^2}\cdot(\vec{\nabla}'\Lambda')\vec{v}$$

$$=[\vec{\nabla}'+\gamma\frac{\vec{v}}{c}(-\frac{1}{c}\frac{\partial}{\partial t'})-(1-\gamma)\frac{\vec{v}}{v^2}\cdot(\vec{\nabla}')\vec{v}]\Lambda' \tag{12}$$

In this time, Eq (3) coincides Eq (11), Eq (12). Therefore a function Λ of S coincides Λ' of S'. Hence $\Lambda=\Lambda'$.

Likely this upper case

$$\phi^2-\vec{A}\cdot\vec{A}=\phi'^2-\vec{A}'\cdot\vec{A}' \tag{13}$$

Eq(13) is used by Eq (7), Eq (8) of the gauge transformation. If we use Eq(3),Eq (5),Eq (6) and Eq(11),Eq(12),

$$(\phi-\frac{1}{c}\frac{\partial\Lambda}{\partial t})^2-(\vec{A}+\vec{\nabla}\Lambda)\cdot(\vec{A}+\vec{\nabla}\Lambda)$$

$$=(\phi^2-\vec{A}\cdot\vec{A})-2(\frac{1}{c}\frac{\partial\Lambda}{\partial t}\phi+\vec{A}\cdot\vec{\nabla}\Lambda)+[\frac{1}{c^2}(\frac{\partial\Lambda}{\partial t})^2-\vec{\nabla}\Lambda\cdot\vec{\nabla}\cdot\Lambda]$$

$$=(\phi'^2-\vec{A}'\cdot\vec{A}')-2(\frac{1}{c}\frac{\partial\Lambda'}{\partial t'}\phi'+\vec{A}'\cdot\vec{\nabla}'\Lambda')+[\frac{1}{c^2}(\frac{\partial\Lambda'}{\partial t'})^2-\vec{\nabla}'\Lambda'\cdot\vec{\nabla}'\cdot\Lambda']$$

$$=(\phi'-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'})^2-(\vec{A}'+\vec{\nabla}'\Lambda')\cdot(\vec{A}'+\vec{\nabla}'\Lambda') \tag{14}$$

$$(\phi^2-\vec{A}\cdot\vec{A})=(\phi'^2-\vec{A}'\cdot\vec{A}'), \tag{15}$$

$$(\frac{1}{c}\frac{\partial\Lambda}{\partial t}\phi+\vec{A}\cdot\vec{\nabla}\Lambda)=(\frac{1}{c}\frac{\partial\Lambda'}{\partial t'}\phi'+\vec{A}'\cdot\vec{\nabla}'\Lambda') \tag{16}$$

$$(\frac{1}{c}\frac{\partial\Lambda}{\partial t})^2-\vec{\nabla}\Lambda\cdot\vec{\nabla}\Lambda=(\frac{1}{c}\frac{\partial\Lambda'}{\partial t'})^2-\vec{\nabla}'\Lambda'\cdot\vec{\nabla}'\Lambda',\Lambda=\Lambda' \tag{17}$$

Eq (13) is invariant by Eq(14) about the gauge transformation, Eq (7),Eq (8). .Eq (13) is invariant about the special relativistic transformation. Therefore, we can think the invariant electro-magnetic potential $\bar{\phi}$ likely the coordinate system's the invariant time.

$$\bar{\phi}^2=\phi^2-\vec{A}\cdot\vec{A}=\phi'^2-\vec{A}'\cdot\vec{A}' \tag{18}$$

In Eq(2), electro-magnetic potential A^α transform likely the differential coordinate dx^α

$$d\tau^2 = dt^2 - \frac{1}{c^2}(d\vec{x}\cdot d\vec{x}) \tag{19}$$

$$d\tau^2 = dt^2(1-\frac{u^2}{c^2}) \ , \ \frac{d\vec{x}}{dt} = \vec{u} \tag{20}$$

$$\bar{\phi}^2 = \phi^2(1-\frac{u^2}{c^2}), \ \bar{\phi} = \phi\sqrt{1-\frac{u^2}{c^2}} \tag{21}$$

$$\frac{\vec{A}}{\phi} = \frac{\vec{u}}{c}, \ \vec{A} = \frac{\vec{u}}{c}\phi \tag{22}$$

An example of Eq (22)'s potential is Lienard-Wiechert potential that made by moving point charge.[3,4]

In the inertial coordinate system and the other inertial coordinate system,

$$\bar{\phi}^2 = \phi^2(1-\frac{u^2}{c^2}) = \phi'^2(1-\frac{u'^2}{c^2}), \ \frac{d\vec{x}}{dt} = \vec{u}, \ \frac{d\vec{x}'}{dt'} = \vec{u}'$$

$$\frac{\vec{A}}{\phi} = \frac{\vec{u}}{c}, \ \vec{A} = \frac{\vec{u}}{c}\phi, \quad \frac{\vec{A}'}{\phi'} = \frac{\vec{u}'}{c}, \ \vec{A}' = \frac{\vec{u}'}{c}\phi'$$

$$d\vec{x} = d\vec{x}' + \gamma\frac{\vec{v}}{c}cdt' - (1-\gamma)\frac{\vec{v}\cdot d\vec{x}'}{v^2}\vec{v}, \ dt = \gamma(dt' + \frac{\vec{v}}{c^2}\cdot d\vec{x}')$$

$$\vec{u} = \frac{d\vec{x}}{dt} = \frac{1}{\gamma}\frac{\vec{u}' + \gamma\vec{v} - (1-\gamma)\frac{\vec{v}\cdot\vec{u}'}{v^2}\vec{v}}{(1+\frac{\vec{v}}{c^2}\cdot\vec{u}')}, \ \vec{u}' = \frac{d\vec{x}'}{dt'}, \phi = \gamma(\phi' + \frac{\vec{v}}{c}\cdot\vec{A}')$$

$$\vec{A} = \frac{\vec{u}}{c}\phi = \frac{1}{\gamma}[\frac{\vec{u}' + \gamma\vec{v} - (1-\gamma)\frac{\vec{v}\cdot\vec{u}'}{v^2}\vec{v}}{(1+\frac{\vec{v}}{c^2}\cdot\vec{u}')}]\cdot\gamma\frac{1}{c}[\phi' + \frac{\vec{v}}{c}\cdot\vec{A}') \ , \ \vec{A}' = \frac{\vec{u}'}{c}\phi'$$

$$= \frac{1}{\gamma}[\frac{\vec{u}' + \gamma\vec{v} - (1-\gamma)\frac{\vec{v}\cdot\vec{u}'}{v^2}\vec{v}}{(1+\frac{\vec{v}}{c^2}\cdot\vec{u}')}]\cdot\gamma\frac{1}{c}\phi'[1 + \frac{\vec{v}}{c}\cdot\frac{\vec{u}'}{c}]$$

$$= \frac{\vec{u}'}{c}\phi' + \gamma\vec{v}\frac{\phi'}{c} - (1-\gamma)\frac{\vec{v}}{v^2}\cdot(\vec{u}'\frac{\phi'}{c})\vec{v}, \vec{A}' = \frac{\vec{u}'}{c}\phi'$$

$$= \vec{A}' + \gamma\frac{\vec{v}}{c}\phi' - (1-\gamma)\frac{\vec{v}\cdot\vec{A}'}{v^2}\vec{v} \quad , \gamma = 1/\sqrt{1-\frac{v^2}{c^2}} \tag{23}$$

According to Eq (22), electro-magnetic field is

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{c\partial t} = -\vec{\nabla}\phi - \frac{\partial}{c\partial t}(\frac{\vec{u}}{c}\phi) = -\vec{\nabla}\phi - \frac{\phi}{c}\frac{\partial\vec{u}}{c\partial t} - \frac{\vec{u}}{c}\frac{\partial\phi}{c\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\frac{\vec{u}}{c}\phi) = \phi(\vec{\nabla}\times\frac{\vec{u}}{c}) - \frac{\vec{u}}{c}\times\vec{\nabla}\phi \tag{24}$$

Electro-magnetic potential equation is in Eq (2),

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\phi = \frac{4\pi}{c}j^0 = 4\pi\rho = \frac{4\pi}{c}\rho_0\frac{dx^0}{d\tau} = \frac{4\pi}{c}\rho_0\frac{cdt}{d\tau} = 4\pi\bar{\gamma}\rho_0$$

$$j^0 = c\rho \;,\; \rho = \bar{\gamma}\rho_0 \;,\; \bar{\gamma} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \tag{25}$$

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\vec{A} = (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\phi\frac{\vec{u}}{c}$$

$$= \frac{\vec{u}}{c}(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\phi + \phi(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\frac{\vec{u}}{c}, (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\frac{\vec{u}}{c} = \vec{0}$$

$$= \frac{\vec{u}}{c}(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\phi, (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\phi = 4\pi\rho$$

$$= \frac{4\pi}{c}\rho\vec{u} = \frac{4\pi}{c}\vec{j}$$

$$\vec{A} = \frac{\vec{u}}{c}\phi \;,\; \vec{j} = \rho\vec{u} \;, \rho = \bar{\gamma}\rho_0 \;,\; \bar{\gamma} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \tag{26}$$

By Eq(21), electro-magnetic invariant potential $\bar{\phi}$ is

$$\bar{\gamma}\bar{\phi} = \phi,\; \bar{\gamma} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \tag{27}$$

Therefore by Eq (25) and Eq (27), electro-magnetic invariant potential $\bar{\phi}$'s equation is

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\bar{\gamma}\bar{\phi} = 4\pi\bar{\gamma}\rho_0$$

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\bar{\phi} = 4\pi\rho_0 \tag{28}$$

3. CONCLUSION

About the gauge functions Λ and Λ', if we use Eq (17),

$$(\frac{1}{c}\frac{\partial\Lambda}{\partial t})^2 - \vec{\nabla}\Lambda\cdot\vec{\nabla}\Lambda = (\frac{1}{c}\frac{\partial\Lambda'}{\partial t'})^2 - \vec{\nabla}'\Lambda'\cdot\vec{\nabla}'\Lambda',\;\; \Lambda = \Lambda' \tag{29}$$

Lorentz gauge is

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0 \;,\; \frac{1}{c}\frac{\partial\phi'}{\partial t'} + \vec{\nabla}'\cdot\vec{A}' = 0 \tag{30}$$

Therefore the gauge functions Λ and Λ' satisfy Eq(31)

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)\Lambda = (\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \nabla'^2)\Lambda' = 0\,(31),\;\; \Lambda = \Lambda' \tag{31}$$

References

[1] A.Miller,Albert Einstein's Special Theory of Relativity(Addison-Wesley Publishing Co., Inc., 1981)

[2] W.Rindler,Special Relativity(2nd ed., Oliver and Boyd,Edinburg, 1966)

[3] David J.Griffiths,Introduction to Electrodynamics(2nd ed.,Prentice Hall)

[4] John R.Reitz,Frederick J.Milford,Robert W.Christy,Foundations of ElectroMagnetic theory(3th ed,.)

[5] William H.Hayt,Jr,Engineering electromagnetics(5th ed.,Mcgraw-hill,1990)

Citation: *Sangwha-Yi, (2019). Gauge Theory's Expansion in the Electro-Magnetic Field. International Journal of Advanced Research in Physical Science (IJARPS) 6(12), pp.1-5, 2019.*